PORTFOLIO

Ing. Tomáš Rábek
tomas.rabek@fem.uniag.sk

Exercise 1

Suppose you had bought a share of General Motor (GM) at the December 1989 for $ 42,25 and sold it a year later, at the end of December 1990, for $ 34,375. During this year, GM paid a per-share dividend of $ 3. Your return from holding GM throughout 1990 would have been

$$r_{GM,1990} = \frac{P_{GM,1990} + Div_{GM,1990} - P_{GM,1989}}{P_{GM,1989}} = \frac{34{,}375 + 3{,}00 - 42{,}25}{42{,}25} = -11{,}54\%$$

The average return over the decade is 14,25 % per year. This number is also called the mean return. We often use the past return to predict future returns, then we also call the mean

the expected return.

$$meanGMreturn = E(r_{GM}) = \bar{r}_{GM} = \frac{r_{GM,1990} + r_{GM,1991} + \ldots\ldots + r_{GM,1999}}{10} = 14{,}25\%$$

Basic Statistics for Asset Returns: Mean, Standard Deviation, Covariance, and Correlation

	A	B	C	D	E
1	**PRICE AND DIVIDEND DATA FOR GENERAL MOTORS (GM)**				
2	**Date**	**Closing Price**	**Dividend**	**Annual return**	
3	29-Dec-89	42,2500	-		
4	31-Dec-90	34,3750	3,00	-11,54%	<-- =(C4+B4)/B3-1
5	31-Dec-91	28,8750	1,60	-11,35%	<-- =(C5+B5)/B4-1
6	31-Dec-92	32,2500	1,40	16,54%	
7	31-Dec-93	54,8750	0,80	72,64%	
8	30-Dec-94	42,1250	0,80	-21,78%	
9	29-Dec-95	52,8750	1,10	28,13%	
10	31-Dec-96	55,7500	1,60	8,46%	
11	31-Dec-97	60,7500	5,59	19,00%	
12	31-Dec-98	71,5625	2,00	21,09%	
13	31-Dec-99	72,6875	14,15	21,34%	
14					
15	Average return, $E(r_{GM})$			14,25%	<-- =AVERAGE(D4:D13)
16	Variance of return, σ^2_{GM}			0,0638	<-- =VARP(D4:D13)
17	Standard deviation of return, σ_{GM}			25,25%	<-- =STDEVP(D4:D13)

Exercise 1

The variance of the annual return is 6,38 %. Variance and standard deviation are statistical measures of the variability of the returns. The variance is calculated with the Excel function = Varp.

$$Var(r_{GM}) = \sigma^2{}_{GM} = \frac{\left(r_{GM,1990} - \bar{r}_{GM}\right)^2 + .\left(r_{GM,1991} - \bar{r}_{GM}\right)^2 + ... + \left(r_{GM,1999} - \bar{r}_{GM}\right)^2}{10} = 6,38\%$$

The standard deviation of the annual returns is the square root of the variance. The standard deviation is calculated with the Excel function directly = Stdevp

$$\sqrt{0,0638} = 25,25\%$$

Covariance and correlation – that relate the return of two stocks to each other.

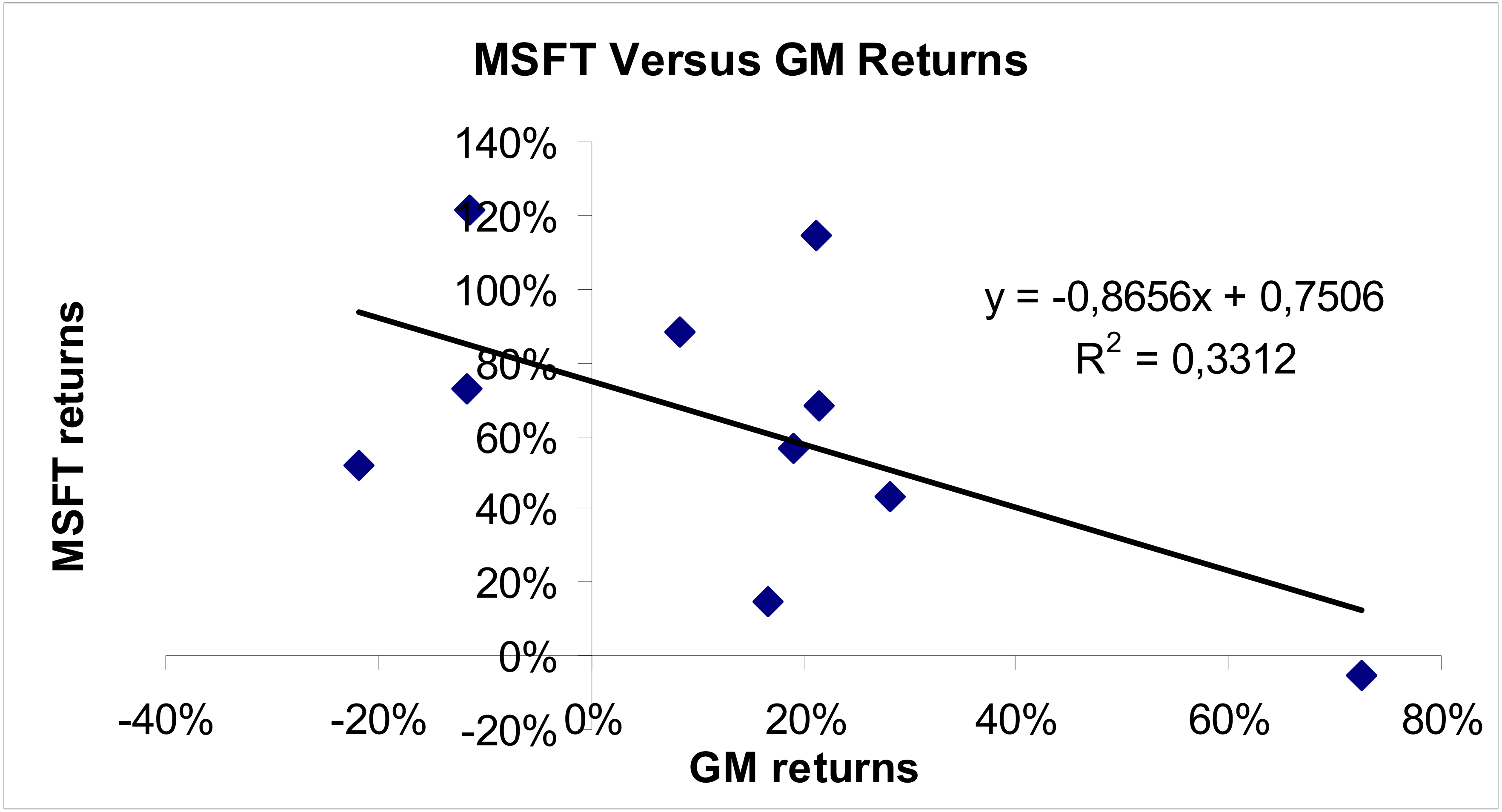
	A	B	C	D
1	**GM AND MSFT, ANNUAL RETURN DATA**			
2	**Date**	**GM return**	**MSFT return**	
3	31-Dec-90	-11,54%	72,99%	
4	31-Dec-91	-11,35%	121,76%	
5	31-Dec-92	16,54%	15,11%	
6	31-Dec-93	72,64%	-5,56%	
7	30-Dec-94	-21,78%	51,63%	
8	29-Dec-95	28,13%	43,56%	
9	31-Dec-96	8,46%	88,32%	
10	31-Dec-97	19,00%	56,43%	
11	31-Dec-98	21,09%	114,60%	
12	31-Dec-99	21,34%	68,36%	
13				
14	Average return, $E(r_{GM})$ and $E(r_{MSFT})$	14,25%	62,72%	
15	Variance of return, σ^2_{GM} and σ^2_{MSFT}	6,38%	14,43%	
16	Standard deviation of return, σ_{GM} and σ_{MSFT}	25,25%	37,99%	
17	Covariance of returns, $Cov(r_{GM},r_{MSFT})$	-0,0552		<-- =COVAR(B3:B12;C3:C12)
18	Correlation of returns, $\rho_{GM,MSFT}$	-0,5755		<-- =CORREL(B3:B12;C3:C12)
19		-0,5755		<-- =B17/(B16*C16)

The covariance between two series is a measure of how much the series move up or down together. Excel has a function **Covar**, which when applied directly to the return, calculates the covariance. The formal definition is

$$Cov(r_{GM}, r_{MSFT}) = \sigma_{GM,MSFT} = \frac{1}{10}\left[\begin{array}{l} (r_{GM,1} - \bar{r}_{GM})*(r_{MSFT,1} - \bar{r}_{MSFT}) + (r_{GM,2} - \bar{r}_{GM})*(r_{MSFT,2} - \bar{r}_{MSFT}) \\ + \ldots + (r_{GM,10} - \bar{r}_{GM})*(r_{MSFT,10} - \bar{r}_{MSFT}) \end{array} \right]$$

The correlation coefficient is always between -1 and + 1. Roughly speaking, two sets of return that have a correlation coefficient of -1 vary perfectly inversely (when one return goes up, we can perfectly predict how the other return goes down. A correlation coefficient of + 1 means that the returns vary in perfect tandem. A correlation coefficient between -1 and + 1 means that the two sets of returns vary together less than perfectly.

$$Correlation(r_{GM}, r_{MSFT}) = \rho_{GM,MSFT} = \frac{Cov(r_{GM}, r_{MSFT})}{\sigma_{GM}\sigma_{MSFT}}$$



MSFT Versus GM Returns
140%
120%
100%
80%
60%
40%
20%
0%
-20%
-40%
-20%
0%
20%
40%
60%
80%
y = -0,8656x + 0,7506
R2 = 0,3312
MSFT returns
GM returns

Portfolio Mean and Variance for a Two-Asset Portfolio

A portfolio is a set of stocks or other financial assets. Most people who own stock own more than one stock, they own portfolios of stocks, and the risks they bear relate to the riskiness of their portfolio

Suppose that between 1990 and 1999 you held a portfolio 50 % in GM and 50 % in MSFT.

$$30{,}73\% = 50\% * (-11{,}54\%) + 50\% * 72{,}99\%$$

	A	B	C	D	E	F
1	**CALCULATING PORTFOLIO RETURNS AND THEIR STATISTICS**					
2	Proportion of GM	0,5				
3	Proportion of MSFT	0,5	<-- =1-B2			
4						
5	**Date**	**General Motors GM**	**Microsoft MSFT**		**Portfolio return**	
6	XII.90	-11,54%	72,99%		30,73%	<-- =$B$2*B6+$B$3*C6
7	XII.91	-11,35%	121,76%		55,21%	
8	XII.92	16,54%	15,11%		15,82%	
9	XII.93	72,64%	-5,56%		33,54%	
10	XII.94	-21,78%	51,63%		14,93%	
11	XII.95	28,13%	43,56%		35,84%	
12	XII.96	8,46%	88,32%		48,39%	
13	XII.97	19,00%	56,43%		37,71%	
14	XII.98	21,09%	114,60%		67,85%	
15	XII.99	21,34%	68,36%		44,85%	
16						
17	Mean	14,25%	62,72%		38,49%	<-- =AVERAGE(E6:E15)
18	Variance	6,38%	14,43%		2,44%	<-- =VARP(E6:E15)
19	St. dev.	25,25%	37,99%		15,62%	<-- =STDEVP(E6:E15)
20	Covariance		-0,0552			
21	Correlation		-0,5755			
22						
23	**Direct calculation of portfolio mean and variance**					
24	Portfolio mean, $E(r_p)$	38,49%	<-- =B2*B17+B3*C17			
25	Portfolio variance, $Var(r_p)$	2,44%	<-- =B2^2*B18+B3^2*C18+2*B2*B3*C20			
26	Portfolio st. dev., σ_p	15,62%	<-- =SQRT(B25)			

To calculate the portfolio mean using:

$$portfoliomeanreturn = E(r_p) = x_{GM}E(r_{GM}) + x_{MSFT}E(r_{MSFT})$$

The formula for the portfolio variance is given by

$$portfolio\,variance = Var(r_p) = x^2{}_{GM}Var(r_{GM}) + x^2{}_{MSFT}Var(r_{MSFT}) + 2x_{Gm}x_{MSFT}Cov(r_{GM}, r_{MSFT})$$

In the spreadsheet below we built a table of the portfolio statistics using the formulas. In the table we vary the proportion of GM stock in the portfolio from 0 % to 100 %.

CALCULATING PORTFOLIO RETURNS AND THEIR STATISTICS FROM THE FORMULAS

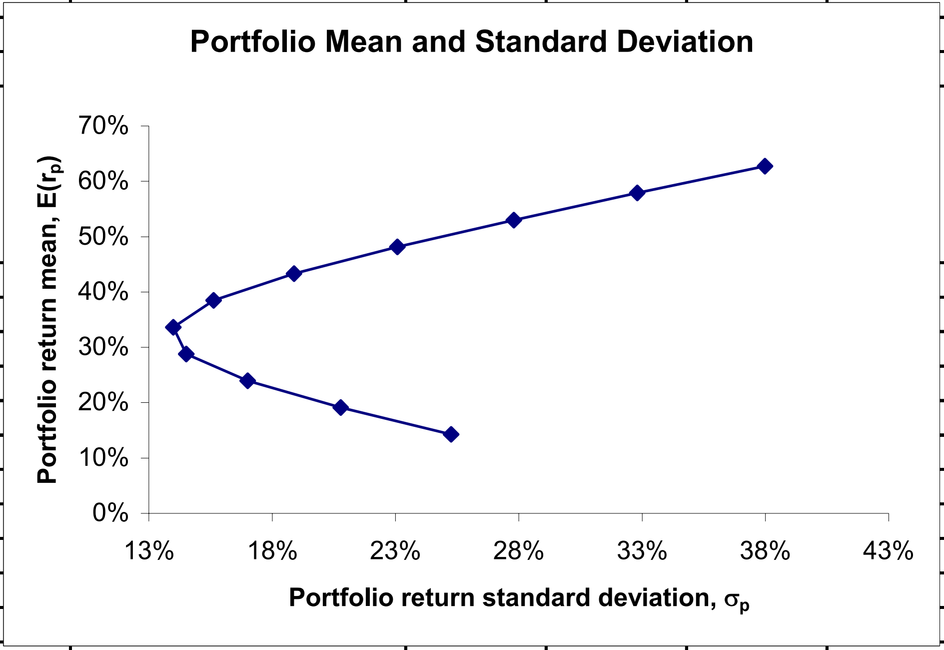
	General Motors GM	Microsoft MSFT
Mean	14,25%	62,72%
Variance	6,38%	14,43%
Standard deviation	25,25%	37,99%
Covariance		-5,52%

Proportion of GM in portfolio	Portfolio Variance $Var(r_p)$	Portfolio standard deviation σ_p	Portfolio mean $E(r_p)$
0%	14,43%	37,99%	62,72%
10%	10,76%	32,80%	57,87%
20%	7,72%	27,79%	53,03%
30%	5,33%	23,08%	48,18%
40%	3,56%	18,88%	43,33%
50%	2,44%	15,62%	38,49%
60%	1,95%	13,98%	33,64%
70%	2,11%	14,51%	28,79%
80%	2,89%	17,01%	23,95%
90%	4,32%	20,78%	19,10%
100%	6,38%	25,25%	14,25%

=A19^2*$B$4+(1-A19)^2*$C$4+2*A19*(1-A19)*$C$6

=SQRT(B19)

=A19*$B$3+(1-A19)*$C$3



Portfolio Statistical for Multiple Assets

The portfolio´s expected return is the weighted average of the individual asset returns. Denoting the portfolio weight by (x1, x2, …, xN) the portfolio expected return is:

$$portfolio mean return = E(r_p) = x_1 E(r_1) + x_2 E(r_2) + ... + x_N E(r_N) = \sum_{i=1}^{N} x_i E(r_i)$$

The portfolio´s variance of return is the sum of the following two expressions:

- The sum of each asset´s variance, weighted by the square of the asset´s portfolio proportion:

$$x^2{}_1 Var(r_1) + x^2{}_2 Var(r_2) + ... + x^2{}_N Var(r_N)$$

- The sum of twice each of the covariance, weighted by the product of the asset proportion:

$$2x_1x_2Cov(r_1,r_2) + 2x_1x_3Cov(r_1,r_3) + ... + 2x_1x_NCov(r_1,r_N) + 2x_2x_3Cov(r_2,r_3) + ... +$$
$$2x_2x_NCov(r_2,r_N) + ... + 2x_{N-1}x_NCov(r_{N-1},r_N)$$

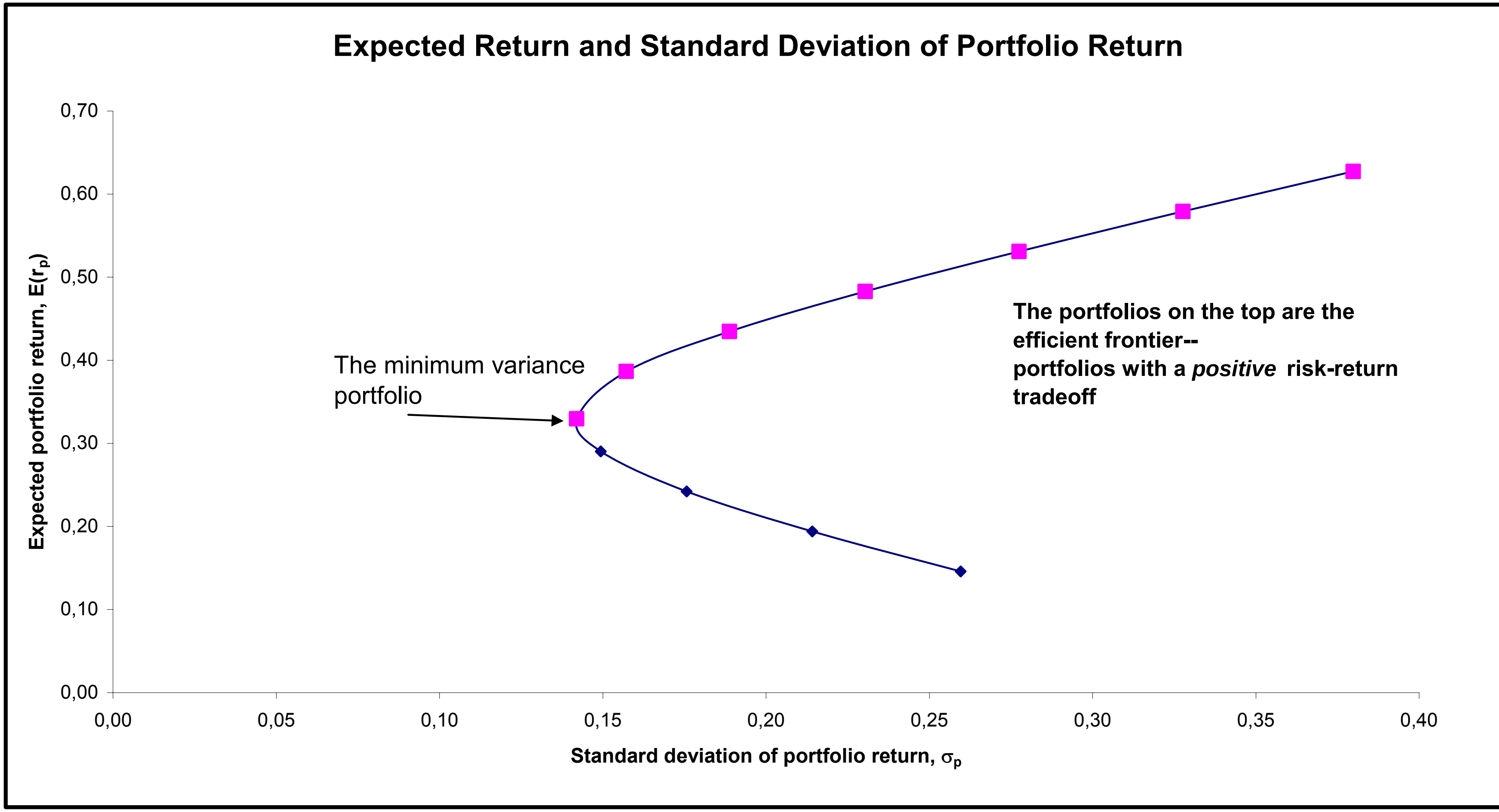
The Efficient Frontier and the Minimum Variance Portfolio

The parabolic shape of the graph is the subject of much discussion in finance.

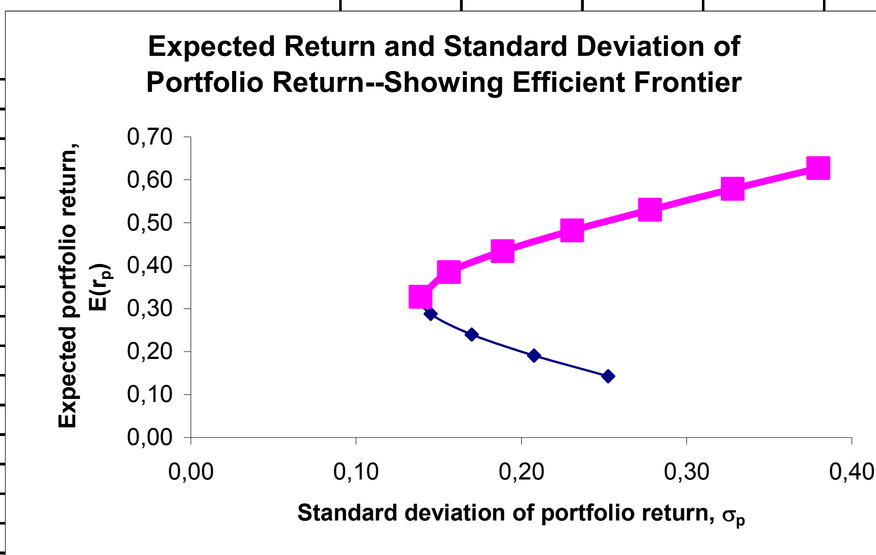
All the portfolios on the top part of the graph have this property. This top part of the graph is called the efficient frontier.

The efficient frontier is the area of hard portfolio choices – along the efficient frontier, portfolios with greater expected return require you to undertake greater risk.

The efficient frontier slopes upward from left to right.


Expected Return and Standard Deviation of Portfolio Return
0,70
0,60
0,50
0,40
0,30
0,20
0,10
0,00
Expected portfolio return, E(r_p)
0,00
0,05
0,10
0,15
0,20
0,25
0,30
0,35
0,40
Standard deviation of portfolio return, σ_p
The minimum variance portfolio
The portfolios on the top are the efficient frontier-- portfolios with a positive risk-return tradeoff

	A	B	C	D	E	F
1	**CALCULATING THE MINIMUM VARIANCE PORTFOLIO WITH A FORMULA**					
2			**GM**	**MSFT**		
3	Average, $E(r_{GM})$ and $E(r_{MSFT})$		14,25%	62,72%		
4	Variance, $Var(r_{GM})$ and $Var(r_{MSFT})$		6,38%	14,43%		
5	Sigma, σ_{GM} and σ_{MSFT}		25,25%	37,99%		
6	Covariance of returns, $Cov(r_{GM},r_{MSFT})$		-5,52%			
7						
8	**Minimum variance portfolio--analytic formula**					
9	Percentage in GM	62,64%	<-- =(D4-C6)/(C4+D4-2*C6)			
10	Percentage in MSFT	37,36%	<-- =1-B9			
11						
12	Expected portfolio return, $E(r_p)$	32,36%	<-- =B9*C3+B10*D3			
13	Portfolio variance, $Var(r_p)$	0,0193	<-- =B9^2*C4+B10^2*D4+2*B9*B10*C6			
14	Portfolio standard deviation, σ_p	13,90%	<-- =SQRT(B13)			



	H	I	J	K
15	**Percentage in GM**	**Sigma**	**Expected return**	**Efficient frontier points**
16				
17	0,00%	37,99%	62,72%	62,72%
18	10,00%	32,80%	57,87%	57,87%
19	20,00%	27,79%	53,03%	53,03%
20	30,00%	23,08%	48,18%	48,18%
21	40,00%	18,88%	43,33%	43,33%
22	50,00%	15,62%	38,49%	38,49%
23	61,847400%	13,91%	32,74%	32,74%
24	70,00%	14,51%	28,79%	
25	80,00%	17,01%	23,95%	
26	90,00%	20,78%	19,10%	
27	1	25,25%	14,25%	
28				
29				
30				
31				
32				

This is the portfolio percentage in GM which gives the minimum variance portfolio.

In this section we examine the effect of stock return correlation on portfolio returns:

- **When the two coins have a perfectly negative correlation of -1, we can create a risk-free asset using combinations of the two assets.**
- **When the two coins have a perfectly positive correlation of +1, it is impossible to diversity away any risk.**
- **When the two coins have correlation between – 1 and + 1, some of the risk can be eliminated through diversification.**

The Capital Asset Pricing Model (CAPM) and The Security Market Line (SML)

In this chapter we discuss two powerful results about returns and risks in capital markets.

- One result, termed the capital market line (CML), gives investors advice about how to invest. The CML says that the best investment portfolio for any investor is a combination of two assets – a risk-free asset such as a savings account and a risk asset representative of the risks of the overall stock market.

- A second result, called the security market line (SML), links the return of any asset to its risk. The SML states that the expected return of any asset depends on the asset´s sensitivity to tho market. This sensitivity is termed beta and is often written with the Greek letter β. Assets with higher β have higher risks and will earn higher expected returns.

The Capital Market Line (CML)

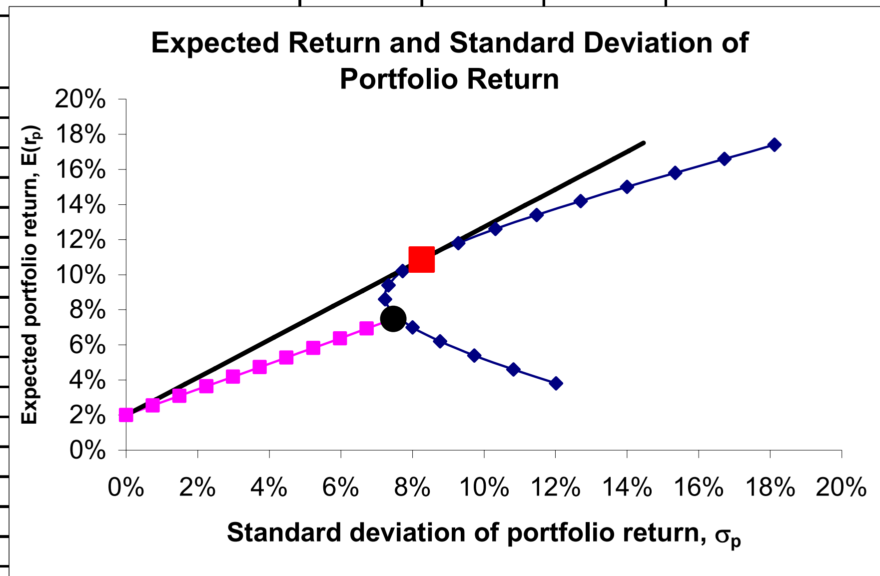
Capital Market Line (CML) says that an investor's optimal investment strategy is to split his/her capital between two assets: a risk-free asset earning rf and a risky asset representing the risks of the overall market.

CML: Expected return of an optimal portfolio

$$E(r_p) = r_f + (\%investedinmarketportfolio) * \lfloor E(r_M) - r_f \rfloor$$

TWO STOCKS AND A RISK-FREE ASSET
The best big square portfolio ②

	Stock A	Stock B	Risk-free r_f
Average return	7,00%	15,00%	2%
Variance	0,0064	0,0196	
Sigma	8,00%	14,00%	
Covariance of returns	0,0011		
Correlation	0,1000		



Round dot portfolio ●		
A	0,9	
B	0,1	
Mean	7,80%	<-- =B29*$B$3+(1-B29)*$C$3
Sigma	7,47%	<-- =SQRT(B28^2*$B$4+(1-B28)^2*$C$4+2*B28*(1-B28)*$B$6)
Best big square portfolio ②		
A	51,81%	
B	48,19%	
Mean	10,85%	<-- =B34*$B$3+(1-B34)*$C$3
Sigma	8,26%	<-- =SQRT(B34^2*$B$4+(1-B34)^2*$C$4+2*B34*(1-B34)*$B$6)

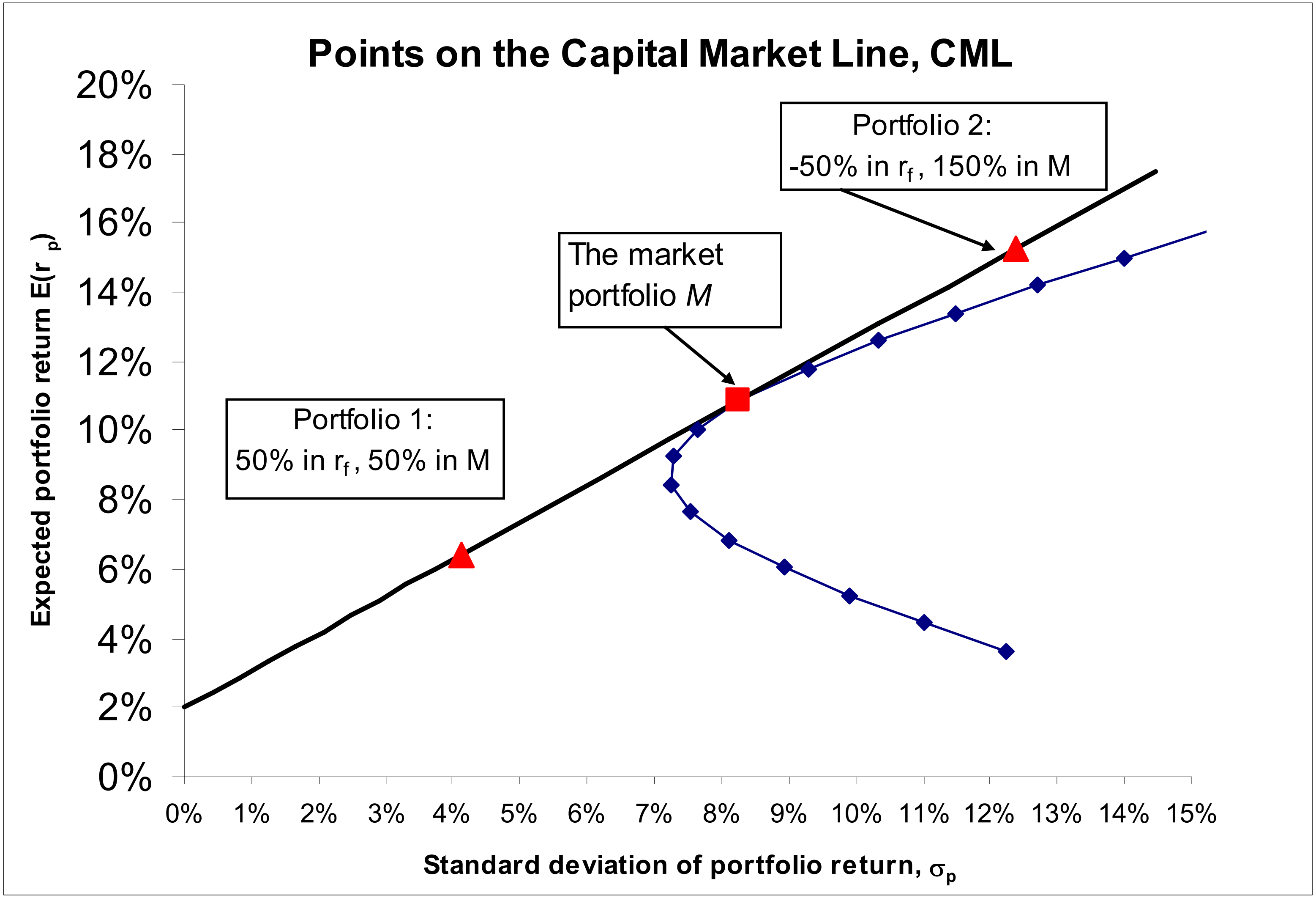
Three Points on the Capital Market Line (CML)

The big square portfolio is called the market portfolio. The market portfolio is the portfolio of risky assets which allows investors to achieve maximal returns. Notice that the line is above the efficient frontier everywhere (except at the point of tangency, which we now call the market portfolio M). We call this line the Capital Market Line (CML).

$$E(r_p) = xE(r_M) + (1 - x)r_f$$

$$\sigma_p = x\sigma_M$$

Three Points on the Capital Market Line (CML)



The Sharpe Ratio and Market Portfolio M

In this section we show to compute the market portfolio M. A concept called the Sharpe ratio – this is on of the standard return/risk measures used in capital market. The portfolio M is the portfolio that maximizes the **Sharpe ratio**.

The portfolio´s risk premium (sometimes called the portfolio excess return) is defined as the difference between its expected return and the return of the risk-free asset:

portfolio´s risk premium = portfolio expected return – risk free rate = E(rp)-rf

The ratio of this risk premium to the portfolio´s standard deviation is called the Sharpe ratio:

$$Sharperatio = \frac{E(r_p) - r_f}{\sigma_p}$$

	A	B	C	D	E
1	**PORTFOLIO RETURNS WITH A RISK-FREE ASSET THE SHARPE RATIO**				
2		**Stock A**	**Stock B**	**Risk-free r_f**	
3	Average return	7,00%	15,00%	2,00%	
4	Variance of return	0,64%	1,96%		
5	Sigma of return	8,00%	14,00%		
6	Covariance of returns	0,0011			
7					
8	**Portfolio return and risk**				
9	Percentage in stock A	51,81%			
10	Percentage in stock B	48,19%			
11					
12	Expected portfolio return	10,85%	<-- =B9*B3+B10*C3		
13	Portfolio standard deviation	8,26%	<-- =SQRT(B9^2*B4+B10^2*C4+2*B9*B10*B6)		
14					
15	Risk premium	8,85%	<-- =B12-D3		
16					
17	Sharpe ratio	1,0716	<-- =(B12-D3)/B13		
18					
19	**The portfolio illustrated has the *highest* Sharpe ratio**				

The Security Market Line (SML)

The SML says that the expected **return of any stock or portfolio** is related to three factors:

1. The risk-free rate in the market rf .

2. The stock's market risk. A stock's risk is measured by a number called beta (β), which measures the sensitivity of the stock's return to the return of the market. If a stock has a high beta, then when the market goes up, the stock goes up even more (and, of course, the opposite-when the market goes down, the stock goes down even more). The price movements of a low beta stock are less sensitive to variations in the market.

3. The expected return of the market, E (rM).

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3. The expected return of the market, E (rM).

The Security Market Line (SML)

The SML states that the expected return of an asset or portfolio is determined by the asset´ s risk (called β), the risk-free rate, and the portfolio that maximizes the Sharpe ratio.

SML: Expected return of any asset

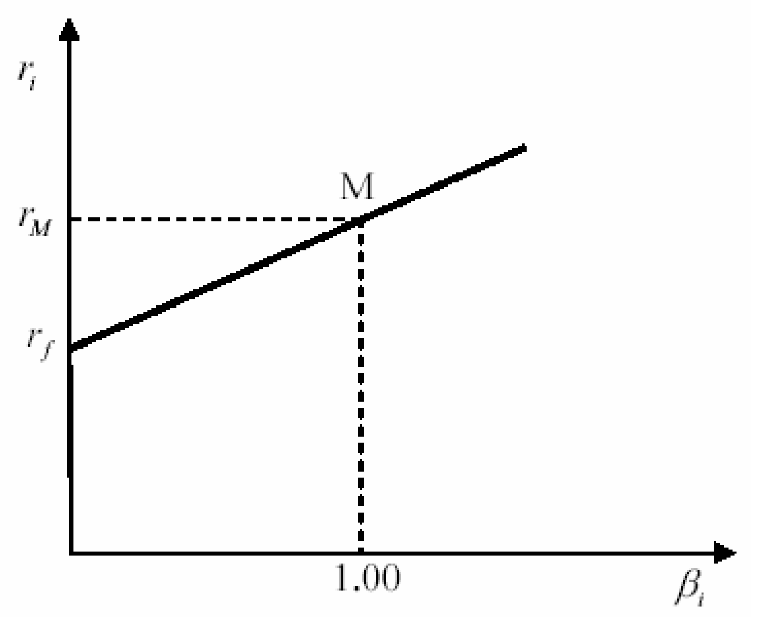
$$E(r_{Asset}) = r_f + \beta_{Asset} * \lfloor E(r_M) - r_f \rfloor$$

E(rM) – is the return on the portfolio that maximizes the Sharpe ratio

$$\beta_i = \frac{Cov(r_i, r_M)}{Var(r_M)}$$

The Security Market Line (SML)

In short, the SML defines the risk/return relation for all assets in the market. The SML is an important tool in investment management, and for computing the cost of capital for a firm.



The end