Return and Risk

9

The previous chapter achieved three purposes. First, we acquainted you with the history of U.S. capital markets. Second, we presented statistics such as expected return, variance, standard deviation, covariance, and correlation. Third, we presented a simplified model of the discount rate on a risky project.

However, we pointed out in the previous chapter that the above model was ad hoc in nature. The next three chapters present a carefully reasoned approach to calculating the discount rate on a risky project. Chapters 9 and 10 examine the risk and the return of individual securities, when these securities are part of a large portfolio. While this investigation is a necessary stepping-stone to discounting projects, corporate projects are not considered here. Rather, a treatment of the appropriate discount rate for capital budgeting is reserved for Chapter 11.

The crux of the current chapter can be summarized as follows: An individual who holds one security should use expected return as the measure of the security's return. Standard deviation or variance is the proper measure of the security's risk. An individual who holds a diversified portfolio cares about the *contribution* of each security to the expected return and the risk of the portfolio. It turns out that a security's expected return is the appropriate measure of the security's contribution to the expected return on the portfolio. Thus, our discussion on expected return in the previous chapter need not be altered, even though portfolios were not explicitly considered at that time. Conversely, neither the security's variance nor the security's standard deviation is an appropriate measure of a security's contribution to the risk of a portfolio. Hence, our previous discussions on risk must be greatly altered to measure a security's contribution to the risk of the entire portfolio of assets.

Portfolios versus Individual Securities 9.1

The previous chapter examined characteristics of individual securities. In particular, we discussed:

1. *Expected return.* This is the return that an individual expects a stock to earn over the next period. Of course, because this is only an expectation, the actual return may be either higher or lower. An individual's expectation may simply be the average return per period a security has earned in the past. Alternatively, it may be based on a detailed analysis of a firm's prospects, on some computer-based model, or on special (or inside) information. Expectations may differ across individuals, though individuals with similar information are likely to share similar expectations.

2. *Variance and standard deviation.* There are many ways to assess the volatility of a security's return. One of the most common is variance, which is a measure of the squared deviations of a security's return from its expected return. Standard deviation, which is the square root of the variance, may be thought of as a standardized version of the variance. Standard deviations and variances for individual securities are generally calculated from past data on returns, though more subjective information may be used as well. As we indicated in Figure 8.10, knowledge of the expected return and the standard deviation can determine the probability that a security's return will fall within a particular range.

3. *Covariance and correlation.* Returns on individual securities are related to one another. Covariance is a statistic measuring the interrelationship between two securities. Alternatively, this relationship can be restated in terms of the correlation between the two securities. The statistical relationship between correlation and covariance is given in equation (8.2). As mentioned earlier, correlation can be viewed as a standardized version of covariance. Although covariance can take on any value, positive or negative, correlation can never be greater than $+1$ or less than -1.

Suppose that an investor has estimates of the expected returns and standard deviations on individual securities and the correlations between securities. How then does the investor choose the best combination or **portfolio** of securities to hold? Obviously, the investor would like a portfolio with a high expected return and a low standard deviation of return. It is therefore worthwhile to consider:

1. The relationship between the expected return on individual securities and the expected return on a portfolio made up of these securities.
2. The relationship between the standard deviations of individual securities, the correlations between these securities, and the standard deviation of a portfolio made up of these securities.

The Example of Supertech and Slowpoke

In order to analyze the above two relationships, we will use the same example of Supertech and Slowpoke that was presented in the previous chapter. The relevant data from the previous chapter are as follows:[1]

[1] See Tables 8.3 and 8.4 for actual calculations.

Relevant Data from Example of Supertech and Slowpoke

Item	Symbol	Value
Expected return on Supertech	$\bar{R}_{Super}$	0.175 = 17.5%
Expected return on Slowpoke	$\bar{R}_{Slow}$	0.055 = 5.5%
Variance of Supertech	σ^2_{Super}	0.066875
Variance of Slowpoke	σ^2_{Slow}	0.013225
Standard deviation of Supertech	σ_{Super}	0.2586 = 25.86%
Standard deviation of Slowpoke	σ_{Slow}	0.1150 = 11.50%
Covariance between Supertech and Slowpoke	$\sigma_{Super,Slow}$	−0.004875
Correlation between Supertech and Slowpoke	$\rho_{Super,Slow}$	−0.1639

The Expected Return on a Portfolio

The formula for expected return on a portfolio is very simple:

The expected return on a portfolio is simply a weighted average of the expected returns on the individual securities.

EXAMPLE

Consider the two securities from the previous chapter, Supertech and Slowpoke. From the above box, we find that the expected returns on the two securities are 17.5 percent and 5.5 percent, respectively.

The expected return on a portfolio of these two securities alone can be written as

$$\text{Expected return on portfolio} = X_{Super}(17.5\%) + X_{Slow}(5.5\%),$$

where X_{Super} is the percentage of the portfolio in Supertech and X_{Slow} is the percentage of the portfolio in Slowpoke. If the investor with $100 invests $60 in Supertech and $40 in Slowpoke, the expected return on the portfolio can be written as

$$\text{Expected return on portfolio} = 0.6 \times 17.5\% + 0.4 \times 5.5\% = 12.7\%$$

Algebraically, we can write

$$\text{Expected return on portfolio} = X_A\bar{R}_A + X_B\bar{R}_B \qquad (9.1)$$

where X_A and X_B are the proportions of the total portfolio in the assets A and B, respectively. (Because our investor can only invest in two securities, $X_A + X_B$ must equal 1 or 100 percent.) $\bar{R}_A$ and $\bar{R}_B$ are the expected returns on the two securities.

Now consider two stocks, each with an expected return of 10 percent. The expected return on a portfolio composed of these two stocks must be 10 percent, regardless of the proportions of the two stocks held. This result may seem obvious at this point, but it will become important later. The result implies that you do not reduce or *dissipate* your expected return by investing in a number of securities. Rather, the expected return on your portfolio is simply a weighted average of the expected returns on the individual assets in the portfolio.

Variance and Standard Deviation of a Portfolio

The Variance

The formula for the variance of a portfolio composed of two securities, A and B, is

The variance of the portfolio:

$$\text{Var(portfolio)} = X_A^2\sigma_A^2 + 2X_AX_B\sigma_{A,B} + X_B^2\sigma_B^2$$

Note that there are three terms on the right-hand side of the equation. The first term involves the variance of A (σ_A^2), the second term involves the covariance between the two securities ($\sigma_{A,B}$), and the third term involves the variance of B (σ_B^2). (As we mentioned in Chapter 8, $\sigma_{A,B} = \sigma_{B,A}$. That is, the ordering of the variables is not relevant when expressing the covariance between two securities.)

The formula indicates an important point. The variance of a portfolio depends on both the variances of the individual securities and the covariance between the two securities. The variance of a security measures the variability of an individual security's return. Covariance measures the relationship between the two securities. For given variances of the individual securities, a positive relationship or covariance between the two securities increases the variance of the entire portfolio. A negative relationship or covariance between the two securities decreases the variance of the entire portfolio. This important result seems to square with common sense. If one of your securities tends to go up when the other goes down, or vice versa, your two securities are offsetting each other. You are achieving what we call a *hedge* in finance, and the risk of your entire portfolio will be low. However, if both your securities rise and fall together, you are not hedging at all. Hence, the risk of your entire portfolio will be higher.

The variance formula for our two securities, Super and Slow, is

$$\text{Var(portfolio)} = X_{\text{Super}}^2\sigma_{\text{Super}}^2 + 2X_{\text{Super}}X_{\text{Slow}}\sigma_{\text{Super,Slow}} + X_{\text{Slow}}^2\sigma_{\text{Slow}}^2 \quad (9.2)$$

Given our earlier assumption that an individual with \$100 invests \$60 in Supertech and \$40 in Slowpoke, $X_{\text{Super}} = 0.6$ and $X_{\text{Slow}} = 0.4$. Using this assumption and the relevant data from the box above, the variance of the portfolio is

$$0.023851 = 0.36 \times 0.066875 + 2 \times (0.6 \times 0.4 \times (-0.004875)) + 0.16 \times 0.013225 \quad (9.2')$$

The Matrix Approach

Alternatively, equation (9.2) can be expressed in the following matrix format:

	Supertech	Slowpoke
Supertech	$X^2_{Super}\sigma^2_{Super}$ $0.024075 = 0.36 \times 0.066875$	$X_{Super}X_{Slow}\sigma_{Super,Slow}$ $-0.00117 = 0.6 \times 0.4 \times (-0.004875)$
Slowpoke	$X_{Super}X_{Slow}\sigma_{Super,Slow}$ $-0.00117 = 0.6 \times 0.4 \times (-0.004875)$	$X^2_{Slow}\sigma^2_{Slow}$ $0.002116 = 0.16 \times 0.013225$

There are four boxes in the matrix. We can add the terms in the boxes to obtain equation (9.2), the variance of a portfolio composed of the two securities. The term in the upper left-hand corner involves the variance of Supertech. The term in the lower right-hand corner involves the variance of Slowpoke. The other two boxes contain the term involving the covariance. These two boxes are identical, indicating why the covariance term is multiplied by 2 in equation (9.2).

At this point, students often find the box approach to be more confusing than equation (9.2). However, the box approach is easily generalized to more than two securities, a task we perform later in this chapter.

Standard Deviation of a Portfolio

Given (9.2′), we can now determine the standard deviation of the portfolio's return. This is

$$\sigma_P = \text{SD (portfolio)} = \sqrt{\text{Var(portfolio)}} = \sqrt{0.023851} = 0.1544 = 15.44\% \tag{9.3}$$

The interpretation of the standard deviation of the portfolio is the same as the interpretation of the standard deviation of an individual security. The expected return on our portfolio is 12.7 percent. A return of −2.74 percent (12.7% − 15.44%) is one standard deviation below the mean and a return of 28.14 percent (12.7% + 15.44%) is one standard deviation above the mean. If the return on the portfolio is normally distributed, a return between −2.74 percent and +28.14 percent occurs about 68 percent of the time.[2]

The Diversification Effect

It is instructive to compare the standard deviation of the portfolio with the standard deviation of the individual securities. The weighted average of the standard deviations of the individual securities is

[2] There are only four equally probable returns for Supertech and Slowpoke, so neither security possesses a normal distribution. Thus, probabilities would be slightly different in our example.

$$\text{Weighted average of standard deviations} = X_{\text{Super}}\,\sigma_{\text{Super}} + X_{\text{Slow}}\sigma_{\text{Slow}}$$
$$0.2012 = 0.6 \times 0.2586 + 0.4 \times 0.115 \qquad (9.4)$$

One of the most important results in this chapter relates to the difference between (9.3) and (9.4). In our example, the standard deviation of the portfolio is *less* than a weighted average of the standard deviations of the individual securities.

We pointed out earlier that the expected return on the portfolio is a weighted average of the expected returns on the individual securities. Thus, we get a different type of result for the standard deviation of a portfolio than we do for the expected return on a portfolio.

It is generally argued that our result for the standard deviation of a portfolio is due to diversification. For example, Supertech and Slowpoke are slightly negatively correlated ($\rho = -0.1639$). Supertech's return is likely to be a little below average if Slowpoke's return is above average. Similarly, Supertech's return is likely to be a little above average if Slowpoke's return is below average. Thus, the standard deviation of a portfolio composed of the two securities is less than a weighted average of the standard deviations of the two securities.

The above example has negative correlation. Clearly, there will be less benefit from diversification if the two securities exhibited positive correlation. How high must the positive correlation be before all diversification benefits vanish?

To answer this question, let us rewrite (9.2) in terms of correlation rather than covariance. We mentioned in Chapter 8 that the covariance can be rewritten as[3]

$$\sigma_{\text{Super,Slow}} = \rho_{\text{Super,Slow}}\sigma_{\text{Super}}\sigma_{\text{Slow}} \qquad (9.5)$$

The formula states that the covariance between any two securities is simply the correlation between the two securities multiplied by the standard deviations of each. In other words, covariance incorporates both (1) the correlation between the two assets and (2) the variability of each of the two securities as measured by standard deviation.

From the previous chapter we know that the correlation between the two securities is -0.1639. Given the variances used in equation (9.2′), the standard deviations are 0.2586 and 0.115 for Supertech and Slowpoke, respectively. Thus, the variance of a portfolio can be expressed as

Variance of the portfolio's return

$$= X^2_{\text{Super}}\sigma^2_{\text{Super}} + 2X_{\text{Super}}X_{\text{Slow}}\rho_{\text{Super,Slow}}\sigma_{\text{Super}}\sigma_{\text{Slow}} + X^2_{\text{Slow}}\sigma^2_{\text{Slow}}$$
$$0.023851 = 0.36 \times 0.066875 + 2 \times 0.6 \times 0.4 \times (-0.1639) \times 0.2586 \times 0.115 + 0.16 \times 0.013225 \qquad (9.6)$$

The middle term on the right-hand side is now written in terms of correlation, ρ, not covariance.

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[3] As with covariance, the ordering of the two securities is not relevant when expressing the correlation between the two securities. That is, $\rho_{\text{Super,Slow}} = \rho_{\text{Slow,Super}}$.

Suppose $\rho_{Super,Slow} = 1$, the highest possible value for correlation. Assume all the other parameters in the example are the same. The variance of the portfolio is

$$\text{Variance of the portfolio's return} = 0.040466 = 0.36 \times 0.066875 + 2 \times (0.6 \times 0.4 \times 1 \times 0.2586 \times 0.115) + 0.16 \times 0.013225$$

The standard deviation is

$$\text{Standard deviation of portfolio's return} = \sqrt{0.040466} = 0.2012 = 20.12\% \qquad (9.7)$$

Note that (9.7) and (9.4) are equal. That is, the standard deviation of a portfolio's return is equal to the weighted average of the standard deviations of the individual returns when $\rho = 1$. Inspection of (9.6) indicates that the variance and hence the standard deviation of the portfolio must drop as the correlation drops below 1. This leads to:

As long as $\rho < 1$, the standard deviation of a portfolio of two securities is *less* than the weighted average of the standard deviations of the individual securities.

In other words, the diversification effect applies as long as there is less than perfect correlation (as long as $\rho < 1$). Thus, our Supertech-Slowpoke example is a case of overkill. We illustrated diversification by an example with negative correlation. We could have illustrated diversification by an example with positive correlation—as long as it was not perfect positive correlation.

Concept Questions

- What are the formulas for the expected return, variance, and standard deviation of a portfolio of two assets?
- What is the diversification effect?

9.2 The Efficient Set for Two Assets

Our results on expected returns and standard deviations are graphed in Figure 9.1. In the figure, there is a dot labelled Slowpoke and a dot labelled Supertech. Each dot represents both the expected return and standard deviation for an individual security. As can be seen, Supertech has both a higher expected return and a higher standard deviation.

The box or "□" in the graph represents a portfolio with 60 percent invested in Supertech and 40 percent invested in Slowpoke. You will recall that we have previously calculated both the expected return and the standard deviation for this portfolio.

The choice of 60 percent in Supertech and 40 percent in Slowpoke is just one of an infinite number of portfolios that can be created. The set of portfolios is sketched by the curved line in Figure 9.2.

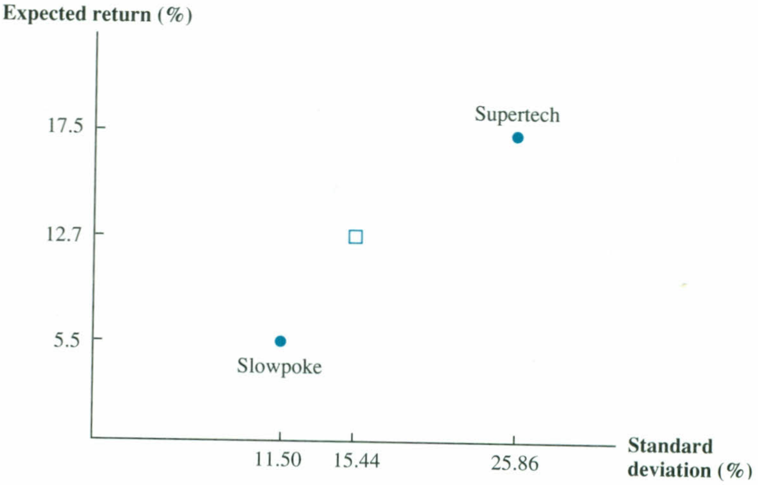

Figure 9.1
Expected return and standard deviation for (1) Supertech, (2) Slowpoke, and (3) a portfolio composed of 60 percent in Supertech and 40 percent in Slowpoke.

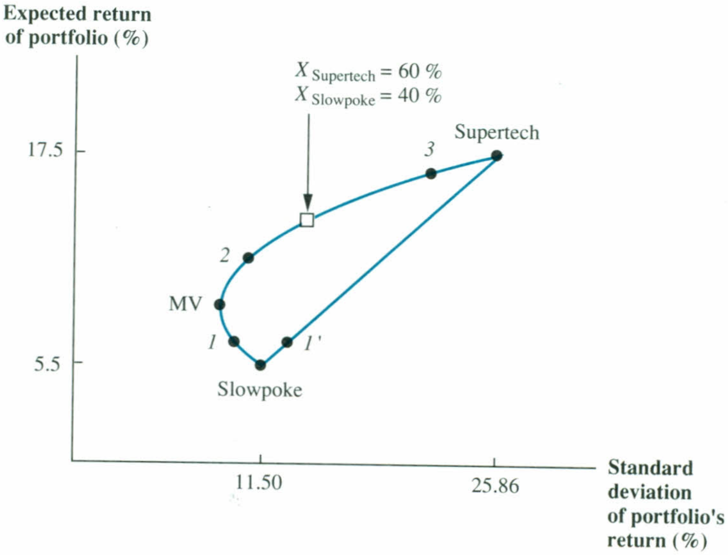

Figure 9.2
Set of portfolios composed of holdings in Supertech and Slowpoke (correlation between the two securities is −0.16).

Portfolio *1* is composed of 90 percent Slowpoke and 10 percent Supertech ($\rho = -0.16$). Portfolio *2* is composed of 50 percent Slowpoke and 50 percent Supertech ($\rho = -0.16$). Portfolio *3* is composed of 10 percent Slowpoke and 90 percent Supertech ($\rho = -0.16$). Portfolio *1′* is composed of 90 percent Slowpoke and 10 percent Supertech ($\rho = 1$). Point MV denotes the minimum variance portfolio. This is the portfolio with the lowest possible variance. By definition, the same portfolio must also have the lowest possible standard deviation.

Consider portfolio *1*. This is a portfolio composed of 90 percent Slowpoke and 10 percent Supertech. Because it is weighted so heavily toward Slowpoke, it appears close to the Slowpoke point on the graph. Portfolio *2* is higher on the curve because it is composed of 50 percent Slowpoke and 50 percent Supertech. Portfolio *3* is close to the Supertech point on the graph because it is composed of 90 percent Supertech and 10 percent Slowpoke.

There are a few important points concerning this graph.

1. We argued that the diversification effect occurs whenever the correlation between the two securities is below 1. The correlation between Supertech and Slowpoke is −0.1639. This diversification effect can be illustrated by comparison with the straight line between the Supertech point and the Slowpoke point. The

straight line represents points that would have been generated had the correlation coefficient between the two securities been 1. The diversification effect is illustrated in the figure since the curved line is always to the left of the straight line. Consider point *1′*. This represents a portfolio composed of 90 percent in Slowpoke and 10 percent in Supertech *if* the correlation between the two were exactly 1. We argue that there is no diversification effect if $\rho = 1$. However, the diversification effect applies to the curved line, because point *1* has the same expected return as point *1′* but has a lower standard deviation. (Points *2′* and *3′* are omitted to reduce the clutter of Figure 9.2.)

Though the straight line and the curved line are both represented in Figure 9.2, they do not simultaneously exist in the same world. *Either* $\rho = -0.1639$ and the curve exists *or* $\rho = 1$ and the straight line exists. In other words, though an investor can choose between different points on the curve if $\rho = -0.1639$, she cannot choose between points on the curve and points on the straight line.

2. The point MV represents the minimum variance portfolio. This is the portfolio with the lowest possible variance. By definition, this portfolio must also have the lowest possible standard deviation. (The term *minimum variance portfolio* is standard in the literature, and we will use that term. Perhaps minimum standard deviation would actually be better, because standard deviation, not variance, is measured on the horizontal axis of Figure 9.2.)

3. An individual contemplating an investment in a portfolio of Slowpoke and Supertech faces an **opportunity set** or **feasible set** represented by the curved line in Figure 9.2. That is, he can achieve any point on the curve by selecting the appropriate mix between the two securities. He cannot achieve any points above the curve because he cannot increase the return on the individual securities, decrease the standard deviations of the securities, or decrease the correlation between the two securities. Neither can he achieve points below the curve because he cannot lower the returns on the individual securities, increase the standard deviations of the securities, or increase the correlation. (Of course, he would not want to achieve points below the curve, even if he were able to do so.)

Were he relatively tolerant of risk, he might choose portfolio *3*. (In fact, he could even choose the end point by investing all his money in Supertech.) An investor with less tolerance for risk might choose point *2*. An investor wanting as little risk as possible would choose MV, the portfolio with minimum variance or minimum standard deviation.

4. Note that the curve is backward bending between the Slowpoke point and MV. This indicates that, for a portion of the feasible set, standard deviation actually decreases as one increases expected return. Students frequently ask, "How can an increase in the proportion of the risky security, Supertech, lead to a reduction in the risk of the portfolio?"

This surprising finding is due to the diversification effect. The returns on the two securities are negatively correlated with each other. One security tends to go up when the other goes down and vice versa. Thus, an addition of a small amount of Supertech acts as a hedge to a portfolio composed only of Slowpoke. The risk of the portfolio is reduced, implying backward bending. Actually, backward bending always occurs if $\rho \leq 0$. It may or may not occur when $\rho > 0$. Of course, the curve bends backward only for a portion of its length. As one continues to increase the

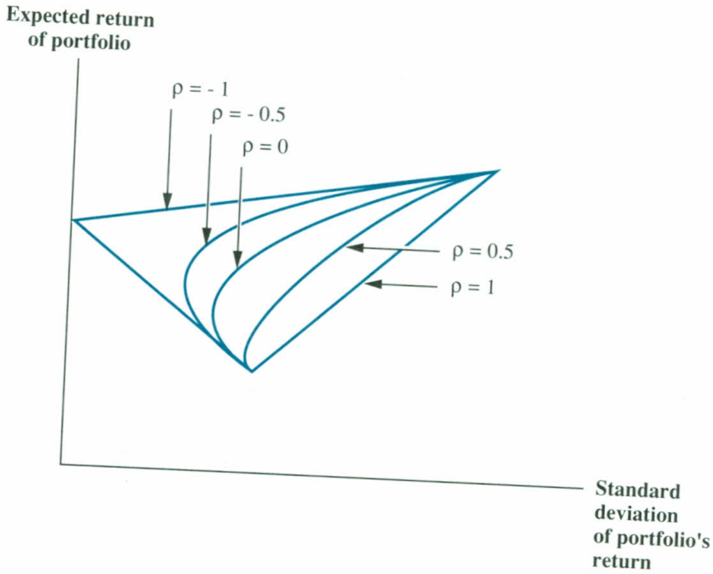

Figure 9.3
Opportunity sets composed of holdings in Supertech and Slowpoke.

Each curve represents a different correlation. The lower the correlation, the more bend in the curve.

percentage of Supertech in the portfolio, the high standard deviation of this security eventually causes the standard deviation of the entire portfolio to rise.

5. No investor would want to hold a portfolio with an expected return below that of the minimum variance portfolio. For example, no investor would choose portfolio *1*. This portfolio has less expected return but more standard deviation than the minimum variance portfolio has. We say that portfolios such as portfolio *1* are *dominated* by the minimum variance portfolio. Though the entire curve from Slowpoke to Supertech is called the feasible set, investors only consider the curve from MV to Supertech. Hence, the curve from MV to Supertech is called the **efficient set.**

Figure 9.2 represents the opportunity set where $\rho = -0.1639$. It is worthwhile to examine Figure 9.3, which shows different curves for different correlations. As can be seen, the lower the correlation, the more bend there is in the curve. This indicates that the diversification effect rises as ρ declines. The greatest bend occurs in the limiting case where $\rho = -1$. This is perfect negative correlation. While this extreme case where $\rho = -1$ seems to fascinate students, it has little practical importance. Most pairs of securities exhibit positive correlation. Strong negative correlation, let alone perfect negative correlation, are unlikely occurrences indeed.[4]

The graphs we examined are not mere intellectual curiosities. Rather, efficient sets can easily be calculated in the real world. As mentioned earlier, data on returns, standard deviations, and correlations are generally taken from past data, though subjective notions can be used to calculate the values of these statistics as well. Once the statistics have been determined, any one of a whole host of software

[4] A major exception occurs with derivative securities. For example, the correlation between a stock and a put on the stock is generally strongly negative. Puts will be treated later in the text.

packages can be purchased to generate an efficient set. However, the choice of the preferred portfolio within the efficient set is up to you. As with other important decisions like what job to choose, what house or car to buy, and how much time to allocate to this course, there is no computer program to choose the preferred portfolio.

An efficient set can be generated where the two individual assets are portfolios themselves. For example, the two assets in Figure 9.4 are a diversified portfolio of American stocks and a diversified portfolio of foreign stocks. Expected returns, standard deviations, and the correlation coefficient were calculated over the period from 1973 to 1988. No subjectivity entered the analysis. The U.S. stock portfolio with a standard deviation of about 0.173 is less risky than the foreign stock portfolio, which has a standard deviation of about 0.222. However, combining a small percentage of the foreign stock portfolio with the U.S. portfolio actually reduces risk, as can be seen by the backward-bending nature of the curve. In other words, the diversification benefits from combining two different portfolios more than offset the introduction of a riskier set of stocks into one's holdings. The minimum variance portfolio occurs with about 80 percent of one's funds in American stocks and about 20 percent in foreign stocks. Addition of foreign securities beyond this point increases the risk of one's entire portfolio.

The backward-bending curve in Figure 9.4 is important information that has not bypassed American money managers. In recent years, pension-fund and mutual-fund managers in the United States have sought out investment opportunities overseas. Another point worth pondering concerns the potential pitfalls of using only past data to estimate future returns. The stock markets of many foreign countries such as Japan have had phenomenal growth in past years. Thus, a graph like Figure 9.4 makes a large investment in these foreign markets seem attractive. However, because abnormally high returns cannot be sustained forever, some subjectivity must be used when forecasting future expected returns.

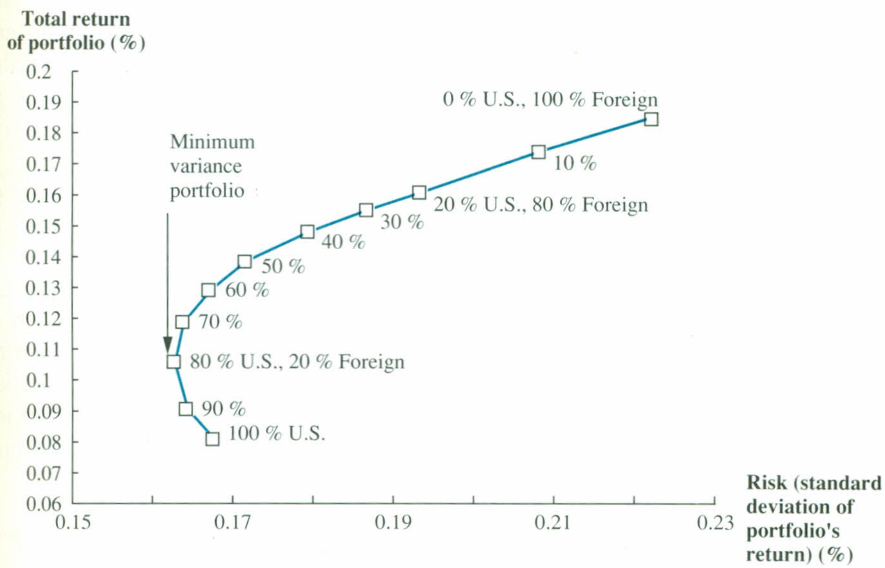

Figure 9.4
Return/risk tradeoff for world stocks: portfolio of U.S. and foreign stocks.

Concept Question

- What is the relationship between the shape of the efficient set for two assets and the correlation between the two assets?

9.3 The Efficient Set for Many Securities

The previous discussion concerned two securities. We found that a simple curve sketched out all the possible portfolios. Because investors generally hold more than two securities, we should look at the same curve when more than two securities are held. The shaded area in Figure 9.5 represents the opportunity set or feasible set when many securities are considered. The shaded area represents all the possible combinations of expected return and standard deviation for a portfolio. For example, in a universe of 100 securities, point 1 might represent a portfolio of, say, 40 securities. Point 2 might represent a portfolio of 80 securities. Point 3 might represent a different set of 80 securities, or the same 80 securities held in different proportions, or something else. Obviously, the combinations are virtually endless. However, note that all possible combinations fit into a confined region. No security or combination of securities can fall outside of the shaded region. That is, no one can choose a portfolio with an expected return above that given by the shaded region because the expected returns on individual securities cannot be altered. Furthermore, no one can choose a portfolio with a standard deviation below that given in the shady area. Perhaps more surprisingly, no one can choose an expected return below that given in the curve. In other words, the capital markets actually prevent a self-destructive person from taking on a guaranteed loss.[5]

So far, Figure 9.5 is different from the earlier graphs. When only two securities are involved, all the combinations lie on a single curve. Conversely, the combinations cover an entire area with many securities. However, notice that an individual will want to be somewhere on the upper edge between *MV* and *X*. The upper edge, which we indicate in Figure 9.5 by a thick line, is called the efficient set. Any point below the efficient set would receive less expected return and the same standard deviation as a point on the efficient set. For example, consider *R* on the efficient set and *W* directly below it. If *W* contains the risk you desire, you should choose *R* instead in order to receive a higher expected return.

In the final analysis, Figure 9.5 is quite similar to Figure 9.2. The efficient set in Figure 9.2 runs from *MV* to Supertech. It contains various combinations of the securities Supertech and Slowpoke. The efficient set in Figure 9.5 runs from *MV* to *X*. It contains various combinations of many securities. The fact that a whole shaded area appears in Figure 9.5 but not in Figure 9.2 is just not an important difference; no investor would choose any point in Figure 9.5 below the efficient set anyway.

We mentioned before that an efficient set for two securities can be traced out easily in the real world. The task becomes more difficult when additional securities are included because the number of observations grows. For example, using

[5] Of course, someone dead set on parting with his money can do so. For example, he can trade frequently without purpose, so that commissions more than offset the positive expected returns on the portfolio.

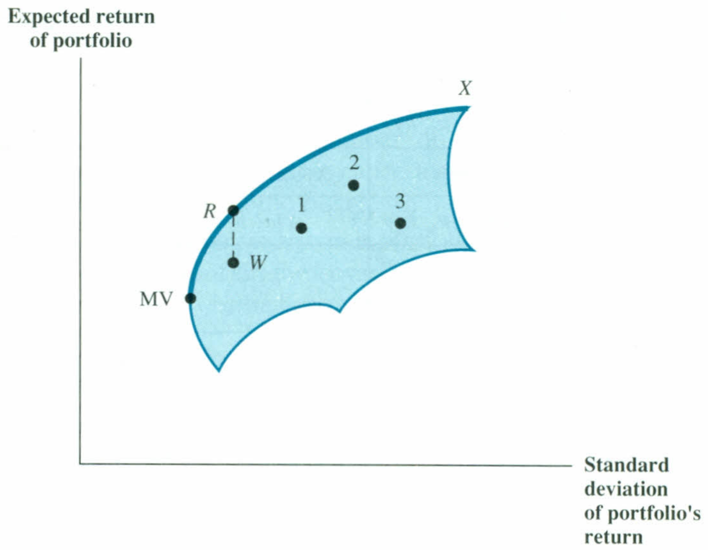

Figure 9.5
The feasible set of portfolios constructed from many securities.

subjective analysis to estimate expected returns and standard deviations for, say, 100 or 500 securities may very well become overwhelming, and the difficulties with correlations may be greater still. There are almost 5,000 correlations between pairs of securities from a universe of 100 securities.

Though much of the mathematics of efficient-set computation had been derived in the 1950s,[6] the high cost of computer time restricted application of the principles. In recent years, the cost has been drastically reduced. A number of software packages allow the calculation of an efficient set for portfolios of moderate size. By all accounts these packages sell quite briskly, so that our above discussion would appear to be important in practice.

Variance and Standard Deviation in a Portfolio of Many Assets

We earlier calculated the formulas for variance and standard deviation in the two-asset case. Because we considered a portfolio of many assets in Figure 9.5, it is worthwhile to calculate the formulas for variance and standard deviation in the many-asset case. The formula for the variance of a portfolio of many assets can be viewed as an extension of the formula for the variance of two assets.

To develop the formula, we employ the same type of matrix that we used in the two-asset case. This matrix is displayed in Table 9.1. Assuming that there are N assets, we write the numbers 1 through N on the horizontal axis and 1 through N on the vertical axis. This creates a matrix of $N \times N = N^2$ boxes.

Consider, for example, the box with a horizontal dimension of 2 and a vertical dimension of 3. The term in the box is $X_3 X_2 \text{Cov}(R_3, R_2)$. X_3 and X_2 are the percentages of the entire portfolio that are invested in the third asset and the

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[6] The classic is Harry Markowitz, *Portfolio Selection* (New York: John Wiley & Sons, 1959).

Table 9.1 Matrix used to calculate the variance of a portfolio

Stock	1	2	3	. . .	N
1	$X_1^2\sigma_1^2$	$X_1X_2\text{Cov}(R_1, R_2)$	$X_1X_3\text{Cov}(R_1, R_3)$		$X_1X_N\text{Cov}(R_1, R_N)$
2	$X_2X_1\text{Cov}(R_2, R_1)$	$X_2^2\sigma_2^2$	$X_2X_3\text{Cov}(R_2, R_3)$		$X_2X_N\text{Cov}(R_2, R_N)$
3	$X_3X_1\text{Cov}(R_3, R_1)$	$X_3X_2\text{Cov}(R_3, R_2)$	$X_3^2\sigma_3^2$		$X_3X_N\text{Cov}(R_3, R_N)$
. . .					
N	$X_NX_1\text{Cov}(R_N, R_1)$	$X_NX_2\text{Cov}(R_N, R_2)$	$X_NX_3\text{Cov}(R_N, R_3)$		$X_N^2\sigma_N^2$

σ_i is the standard deviation of stock i.

$\text{Cov}(R_i, R_j)$ is the covariance between stock i and stock j.

Terms involving the standard deviation of a single security appear on the diagonal. Terms involving covariance between two securities appear off the diagonal.

second asset, respectively. For example, if an individual with a portfolio of $1,000 invests $100 in the second asset, X_2 = 10% ($100/$1,000). $\text{Cov}(R_3, R_2)$ is the covariance between the returns on the third asset and the returns on the second asset. Next, note the box with a horizontal dimension of 3 and a vertical dimension of 2. The term in the box is $X_2 X_3 \text{Cov}(R_2, R_3)$. Because $\text{Cov}(R_3, R_2) = \text{Cov}(R_2, R_3)$, both boxes have the same value. The second security and the third security make up one pair of stocks. In fact, every pair of stocks appears twice in the table, once in the lower left-hand side and once in the upper right-hand side.

Suppose that the vertical dimension equals the horizontal dimension. For example, the term in the box is $X_1^2\sigma_1^2$ when both dimensions are one. Here, σ_1^2 is the variance of the return on the first security.

Thus, the diagonal terms in the matrix contain the variances of the different

Table 9.2 Number of variance and covariance terms as a function of the number of stocks in the portfolio

Number of stocks in portfolio	Total number of terms	Number of variance terms (number of terms on diagonal)	Number of covariance terms (number of terms off diagonal)
1	1	1	0
2	4	2	2
3	9	3	6
10	100	10	90
100	10,000	100	9,900
. . .	. . .	. . .	. . .
N	N^2	N	$N^2 - N$

In a large portfolio, the number of terms involving covariance between two securities is much greater than the number of terms involving variance of a single security.

stocks. The off-diagonal terms contain the covariances. Table 9.2 relates the numbers of diagonal and off-diagonal elements to the size of the matrix. The number of diagonal terms (number of variance terms) is always the same as the number of stocks in the portfolio. The number of off-diagonal terms (number of covariance terms) rises much faster than the number of diagonal terms. For example, a portfolio of 100 stocks has 9,900 covariance terms. Since the variance of a portfolio's returns is the sum of all the boxes, we have:

> The variance of the return on a portfolio with many securities is more dependent on the covariances between the individual securities than on the variances of the individual securities.

Concept Questions

- What is the formula for the variance of a portfolio for many assets?
- How can the formula be expressed in terms of a box or matrix?

Diversification: An Example **9.4**

The above point can be illustrated by altering the matrix in Table 9.1 slightly. Suppose that we make the following three assumptions:

1. All securities possess the same variance, which we write as $\overline{\text{var}}$. In other words, $\sigma_i^2 = \overline{\text{var}}$ for every security.
2. All covariances in Table 9.1 are the same. We represent this uniform covariance as $\overline{\text{cov}}$. In other words, $\text{Cov}(R_i, R_j) = \overline{\text{cov}}$ for every pair of securities. It can easily be shown that $\overline{\text{var}} > \overline{\text{cov}}$.
3. All securities are equally weighted in the portfolio. Because there are N assets, the weight of each asset in the portfolio is $1/N$. In other words, $X_i = 1/N$ for each security i.

Table 9.3 is the matrix of variances and covariances under these three simplifying assumptions. Note that all of the diagonal terms are identical. Similarly, all of the off-diagonal terms are identical. As with Table 9.1, the variance of the portfolio is the sum of the terms in the boxes in Table 9.3. We know that there are N diagonal terms involving variance. Similarly, there are $N \times (N - 1)$ off-diagonal terms involving covariance. Summing across all the boxes in Table 9.3, we can express the variances of the portfolio as

$$\text{Variance of portfolio} = \underbrace{N}_{\substack{\text{Number of}\\\text{diagonal}\\\text{terms}}} \times \underbrace{\left(\frac{1}{N^2}\right)\overline{\text{var}}}_{\substack{\text{Each}\\\text{diagonal}\\\text{term}}} + \underbrace{N(N-1)}_{\substack{\text{Number of}\\\text{off-diagonal}\\\text{terms}}} \times \underbrace{\left(\frac{1}{N^2}\right)\overline{\text{cov}}}_{\substack{\text{Each}\\\text{off-diagonal}\\\text{term}}}$$

$$= \left(\frac{1}{N}\right)\overline{\text{var}} + \left(\frac{N^2 - N}{N^2}\right)\overline{\text{cov}}$$

$$= \left(\frac{1}{N}\right)\overline{\text{var}} + \left(1 - \frac{1}{N}\right)\overline{\text{cov}} \tag{9.8}$$

Table 9.3 Matrix used to calculate the variance of a portfolio when (a) all securities possess the same variance, which we represent as $\overline{\text{var}}$; (b) all pairs of securities possess the same covariance, which we represent as $\overline{\text{cov}}$; (c) all securities are held in the same proportion, which is $1/N$.

Stock	1	2	3	. . .	N
1	$(1/N^2)\overline{\text{var}}$	$(1/N^2)\overline{\text{cov}}$	$(1/N^2)\overline{\text{cov}}$		$(1/N^2)\overline{\text{cov}}$
2	$(1/N^2)\overline{\text{cov}}$	$(1/N^2)\overline{\text{var}}$	$(1/N^2)\overline{\text{cov}}$		$(1/N^2)\overline{\text{cov}}$
3	$(1/N^2)\overline{\text{cov}}$	$(1/N^2)\overline{\text{cov}}$	$(1/N^2)\overline{\text{var}}$		$(1/N^2)\overline{\text{cov}}$
. . .					
N	$(1/N^2)\overline{\text{cov}}$	$(1/N^2)\overline{\text{cov}}$	$(1/N^2)\overline{\text{cov}}$		$(1/N^2)\overline{\text{var}}$

Equation (9.8) expresses the variance of our special portfolio as a weighted sum of the average security variance and the average covariance.[7] The intuition appears when we increase the number of securities in the portfolio without limit. The variance of the portfolio becomes

$$\text{Variance of portfolio (when } N \rightarrow \infty) = \overline{\text{cov}} \tag{9.9}$$

This occurs because (1) the weight on the variance term, $1/N$, goes to 0 as N goes to infinity, and (2) the weight on the covariance term, $1 - 1/N$, goes to 1 as N goes to infinity.

Formula (9.9) provides an interesting and important result. In our special portfolio, the variances of the individual securities completely vanish as the number of securities becomes large. However, the covariance terms remain. In fact, the variance of the portfolio becomes the average covariance, $\overline{\text{cov}}$. One often hears that one should diversify. You should not put all your eggs in one basket. The effect of diversification on the risk of a portfolio can be illustrated in this example. The variances of the individual securities are diversified away, but the covariance terms cannot be diversified away.

The fact that part, but not all, of one's risk can be diversified away should be explored. Consider Mr. Smith, who brings $1,000 to the roulette table at a casino. It would be very risky if he put all his money on one spin of the wheel. For example, imagine that he put the full $1,000 on red at the table. If the wheel showed red, he would get $2,000, but if the wheel showed black, he would lose everything.

[7] Equation (9.8) is actually a weighted *average* of the variance and covariance terms because the weights, $1/N$ and $1 - 1/N$, sum to 1.

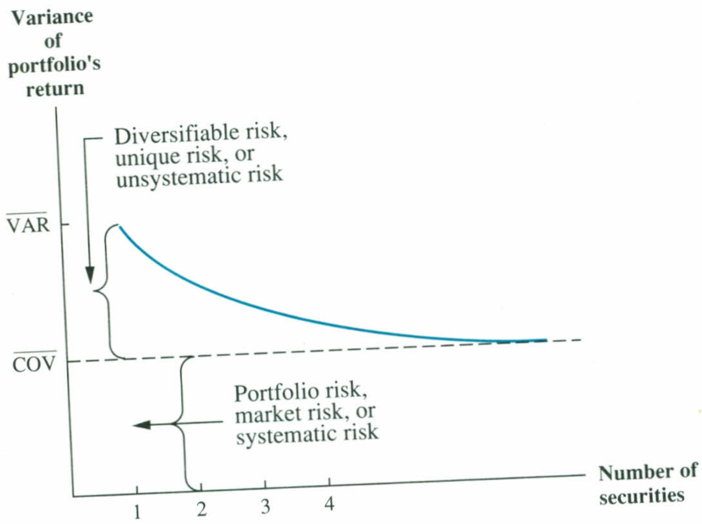

Figure 9.6 Relationship between the variance of a portfolio's return and the number of securities in the portfolio.*

* This graph assumes
a. All securities have constant variance, $\overline{var}$.
b. All securities have constant covariance, $\overline{cov}$.
c. All securities are equallly weighted in portfolio.

The variance of a portfolio drops as more securities are added to the portfolio. However, it does not drop to zero. Rather, $\overline{cov}$ serves as the floor.

Suppose, instead, he divided his money over 1,000 different spins by betting $1 at a time on red. Probability theory tells us that he could count on winning about 50 percent of the time. In other words, he could count on pretty nearly getting all his original $1,000 back.[8]

Now, let's contrast this with our stock-market example, which we illustrate in Figure 9.6. The variance of the portfolio with only one security is, of course, $\overline{var}$ because the variance of a portfolio with one security is the variance of the security. The variance of the portfolio drops as more securities are added, which is evidence of the diversification effect. However, unlike Mr. Smith's roulette example, the portfolio's variance can never drop to zero. Rather it reaches a floor of $\overline{cov}$, which is the covariance of each pair of securities.[9]

Because the variance of the portfolio asymptotically approaches $\overline{cov}$, each additional security continues to reduce risk. Thus, if there were neither commissions nor other transactions costs, it could be argued that one can never achieve too much diversification. However, there is a cost to diversification in the real world. Commissions per dollar invested fall as one makes larger purchases in a single stock. Unfortunately, one must buy fewer shares of each security when buying more and more different securities. Comparing the costs and benefits of diversification, Meir Statman argues that a portfolio of about 30 stocks is needed to achieve optimal diversification.[10]

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[8] This ignores the casino's cut.

[9] Though it is harder to show, this risk reduction effect also applies to the general case where variances and covariances are *not* equal.

[10] Meir Statman, "How Many Stocks Make a Diversified Portfolio?" *Journal of Financial and Quantitative Analysis* (September 1987).

We mentioned earlier that $\overline{\text{var}}$ must be greater than $\overline{\text{cov}}$. Thus, the variance of a security's return can be broken down in the following way:

$$\underset{(\overline{\text{var}})}{\text{Total risk of individual security}} = \underset{(\overline{\text{cov}})}{\text{Portfolio risk}} + \underset{(\overline{\text{var}} - \overline{\text{cov}})}{\text{Unsystematic or diversifiable risk}}$$

Total risk, which is $\overline{\text{var}}$ in our example, is the risk that one bears by holding onto one security only. *Portfolio risk* is the risk that one still bears after achieving full diversification, which is $\overline{\text{cov}}$ in our example. Portfolio risk is often called **systematic** or **market risk** as well. **Diversifiable**, **unique**, or **unsystematic risk** is that risk that can be diversified away in a large portfolio, which must be ($\overline{\text{var}} - \overline{\text{cov}}$) by definition.

To an individual who selects a diversified portfolio, the total risk of an individual security is not important. When considering adding a security to a diversified portfolio, the individual cares about that portion of the risk of a security that cannot be diversified away. This risk can alternatively be viewed as the *contribution* of a security to the risk of an entire portfolio. We will talk later about the case where securities make different contributions to the risk of the entire portfolio.

Risk and the Sensible Investor

Having gone to all this trouble to show that unsystematic risk disappears in a well-diversified portfolio, how do we know that investors even want such portfolios? Suppose they like risk and don't want it to disappear?

We must admit that, theoretically at least, this is possible, but we will argue that it does not describe what we think of as the typical investor. Our typical investor is **risk averse**. Risk-averse behavior can be defined in many ways, but we prefer the following example: A fair gamble is one with zero expected return; a risk-averse investor would prefer to avoid fair gambles.

Why do investors choose well-diversified portfolios? Our answer is that they are risk averse, and risk-averse people avoid unnecessary risk, such as the unsystematic risk on a stock. If you do not think this is much of an answer to why investors choose well-diversified portfolios and avoid unsystematic risk, consider whether you would take on such a risk. For example, suppose you had worked all summer and had saved $5,000, which you intended to use for your college expenses. Now, suppose someone came up to you and offered to flip a coin for the money: heads, you would double your money, and tails, you would lose it all.

Would you take such a bet? Perhaps you would, but most people would not. Leaving aside any moral question that might surround gambling and recognizing that some people would take such a bet, it's our view that the average investor would not.

To induce the typical risk-averse investor to take a fair gamble, you must sweeten the pot. For example, you might need to raise the odds of winning from 50-50 to 70-30 or higher. The risk-averse investor can be induced to take fair gambles only if they are sweetened so that they become unfair to the investor's advantage.

Concept Questions

- What are the two components of the total risk of a security?
- Why doesn't diversification eliminate all risk?

Riskless Borrowing and Lending 9.5

Figure 9.5 assumes that all the securities on the efficient set are risky. Alternatively, an investor could easily combine a risky investment with an investment in a riskless or risk-free security, such as an investment in United States Treasury bills. This is illustrated in the following example.

- **EXAMPLE**

Ms. Logue is considering investing in the common stock of Merville Enterprises. In addition, Ms. Logue will either borrow or lend at the risk-free rate. The relevant parameters are

	Common stock of Merville	Risk-free asset
	Expected return:	Guaranteed return:
Return	14%	10%
Standard deviation:	0.20	0

Suppose Ms. Logue chooses to invest a total of \$1,000, \$350 of which is to be invested in Merville Enterprises and \$650 placed in the risk-free asset. The expected return on her total investment is simply a weighted average of the two returns:

$$\begin{array}{l}\text{Expected return on portfolio}\\ \quad\text{composed of one riskless}\\ \quad\quad\text{and one risky asset}\end{array} = 0.114 = 0.35 \times 0.14 + 0.65 \times 0.10 \qquad \textbf{(9.10)}$$

Because the expected return on the portfolio is a weighted average of the expected return on the risky asset (Merville Enterprises) and the risk-free return, the calculation is analogous to the way we treated two risky assets. In other words, equation (9.1) applies here.

Using equation (9.2), the formula for the variance of the portfolio can be written as

$$X^2_{\text{Merville}}\sigma^2_{\text{Merville}} + 2X_{\text{Merville}}X_{\text{Risk-free}}\sigma_{\text{Merville,Risk-free}} + X^2_{\text{Risk-free}}\sigma^2_{\text{Risk-free}}$$

However, by definition, the risk-free asset has no variability. Thus, both $\sigma_{\text{Merville,Risk-free}}$ and $\sigma^2_{\text{Risk-free}}$ are equal to zero, reducing the above expression to

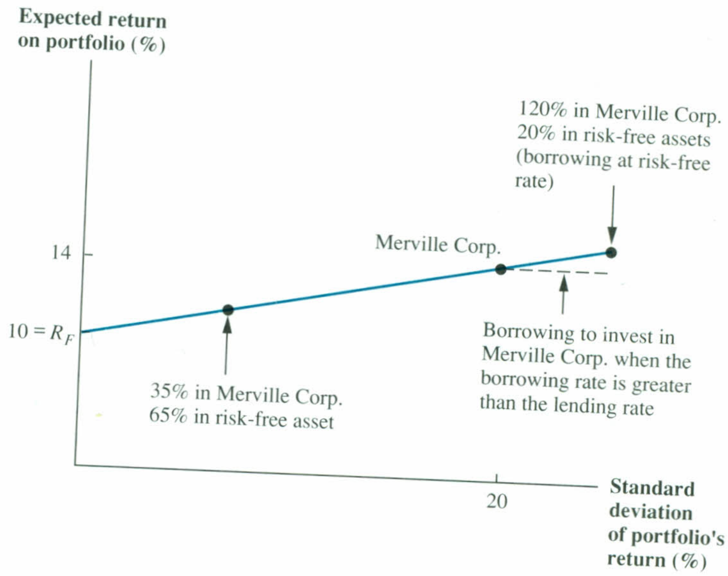

Figure 9.7 Relationship between expected return and risk for a portfolio of one risky asset and one riskless asset.

Variance of portfolio composed of one riskless and one risky asset =

$$X^2_{\text{Merville}}\sigma^2_{\text{Merville}} = (0.35)^2 \times (0.20)^2 = 0.0049 \quad (9.11)$$

The standard deviation of the portfolio is

Standard deviation of portfolio composed of one riskless and one risky asset =

$$X_{\text{Merville}}\sigma_{\text{Merville}} = 0.35 \times 0.20 = 0.07 \quad (9.12)$$

The relationship between risk and return for one risky and one riskless asset can be seen in Figure 9.7. Ms. Logue's split of 35-65 percent between the two assets is represented on a *straight* line between the risk-free rate and a pure investment in Merville Corp. Note that, unlike the case of two risky assets, the opportunity set is straight, not curved.

Suppose that, alternatively, Ms. Logue borrows $200 at the risk-free rate. Combining this with her original sum of $1,000, she invests a total of $1,200 in the Merville Corp. Her expected return would be

Expected return on portfolio formed by borrowing to invest in risky asset $= 14.8\% = 1.20 \times 0.14 + (-0.2) \times 0.10$

Here, she invests 120 percent of his original investment of $1,000 by borrowing 20 percent of her original investment. Note that the return of 14.8 percent is greater than the 14-percent expected return on Merville Corp. This occurs because she is borrowing at 10 percent to invest in a security with an expected return greater than 10 percent.

The standard deviation is

$$\begin{array}{r}\text{Standard deviation of portfolio} \\ \text{formed by borrowing} \\ \text{to invest in risky asset}\end{array} = 0.24 = 1.20 \times 0.2$$

The standard deviation of 0.24 is greater than 0.20, the standard deviation of the Merville Corp., because borrowing increases the variability of the investment. This investment also appears in Figure 9.7.

So far, we have assumed that Ms. Logue is able to borrow at the same rate at which she can lend.[11] Now let us consider the case where the borrowing rate is above the lending rate. The dotted line in Figure 9.7 illustrates the opportunity set for borrowing opportunities in this case. The dotted line is below the solid line because a higher borrowing rate lowers the expected return on the investment.

The Optimal Portfolio

The previous section concerned a portfolio formed between one riskless asset and one risky asset. In reality, an investor is likely to combine an investment in the riskless asset with a portfolio of risky assets. This is illustrated in Figure 9.8.

Consider point *Q*, representing a portfolio of securities. Point *Q* is in the interior of the feasible set of risky securities. Let us assume the point represents a portfolio of 30 percent in AT&T, 45 percent in General Motors (GM) and 25 percent in IBM. Individuals combining investments in *Q* with investments in the riskless asset would achieve points along the straight line from R_F to *Q*. We refer to this as line *I*. For example, point *1* represents a portfolio of 70 percent in the riskless asset and 30 percent in stocks represented by *Q*. An investor with $100 choosing point *1* as his portfolio would put $70 in the risk-free asset and $30 in *Q*. This can be restated as $70 in the riskless asset, $9 (0.3 × $30) in AT&T, $13.50 (0.45 × $30) in GM, and $7.50 (0.25 × $30) in IBM. Point *2* also represents a portfolio of the risk-free asset and *Q*, with more (65%) being invested in Q.

Point *3* is obtained by borrowing to invest in *Q*. For example, an investor with $100 of his own would borrow $40 from the bank or broker in order to invest $140 in *Q*. This can be stated as borrowing $40 and contributing $100 of one's own money in order to invest $42 (0.3 × $140) in AT&T, $63 (0.45 × $140) in GM, and $35 (0.25 × $140) in IBM.

Though any investor can obtain any point on line *I*, no point on the line is optimal. To see this, consider line *II*, a line running from R_F through *A*. Point *A* represents a portfolio of risky securities. Line *II* represents portfolios formed by combinations of the risk-free asset and the securities in *A*. Points between R_F and *A* are portfolios in which some money is invested in the riskless asset and the rest is placed in *A*. Points past *A* are achieved by borrowing at the riskless rate to buy more of *A* than one could with one's original funds alone.

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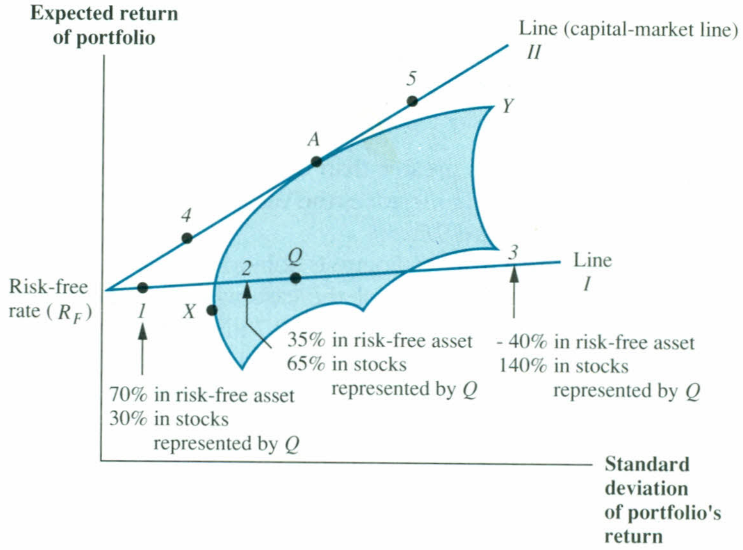
[11] Surprisingly, this appears to be a decent approximation because a large number of investors are able to borrow on margin when purchasing stocks. The borrowing rate on the margin is very near the riskless rate of interest, particularly for large investors. More will be said about this in a later chapter.

Figure 9.8
Relationship between expected return and standard deviation for an investment in a combination of risky securities and the riskless asset.

Portfolio Q is composed of
30 percent AT&T
45 percent GM
25 percent IBM



As drawn, line *II* is tangent to the efficient set of risky securities. Whatever point an individual can obtain on line *I*, he can obtain a point with the same standard deviation and a higher expected return on line *II*. In fact, because line *II* is tangent to the efficient set, it provides the investor with the best possible opportunities. In other words, line *II*, which is frequently called the **capital market line**, can be viewed as the efficient set of *all* assets, both risky and riskless. An investor with a fair degree of risk aversion might choose a point between R_F and A, perhaps point *4*. An individual with less risk aversion might choose a point closer to A or even beyond A. For example, point *5* corresponds to an individual borrowing money to increase his investment in A.

The graph illustrates an important point. With riskless borrowing and lending, the portfolio of ***risky*** assets held by any investor would always be point A. Regardless of the investor's tolerance for risk, he would never choose any other point on the efficient set of risky assets (represented by curve *XAY*) nor any point in the interior of the feasible region. Rather, he would combine the securities of A with the riskless assets if he had high aversion to risk. He would borrow the riskless asset to invest more funds in A had he low aversion to risk.

This result establishes what financial economists call the **separation principle**. That is, the investor makes two separate decisions:

1. After estimating (a) the expected return and variances of individual securities, and (b) the covariances between pairs of securities, the investor calculates the efficient set of risky assets, represented by curve *XAY* in Figure 9.8, and determines Point A, the tangency between the risk-free rate and the efficient set of risky assets (curve *XAY*). Point A represents the portfolio of risky assets that the investor will hold. This point is determined solely from her estimates of returns, variances, and covariances. No personal characteristics, such as degree of risk-aversion, are needed in this step.

2. The investor must now determine how she will combine point A, her portfolio of risky assets, with the riskless asset. She could invest some of her funds in the riskless asset and some in portfolio A. She would end up at a point on the line between R_F and A in this case. Alternatively, she could borrow at the risk-free rate and contribute some of her own fund as well, investing the sum in portfolio A. She would end up at a point on line II beyond A. Her position in the riskless asset, that is, her choice of where on the line she wants to be, is determined by her internal characteristics, such as her ability to tolerate risk.

Concept Questions

- What is the formula for the standard deviation of a portfolio composed of one riskless and one risky asset?
- How does one determine the optimal portfolio among the efficient set of risky assets?

Market Equilibrium 9.6

Definition of the Market-Equilibrium Portfolio

The above analysis concerns one investor. Her estimates of the expected returns and variances for individual securities and the covariances between pairs of securities are hers and hers alone. Other investors would obviously have different estimates of the above variables. However, the estimates might not vary much because all investors would be forming expectations from the same data on past price movement and other publicly available information.

Financial economists often imagine a world where all investors possess the *same* estimates on expected returns, variances, and covariances. Though this can never be literally true, it can be thought of as a useful simplifying assumption in a world where investors have access to similar sources of information. This assumption is called **homogeneous expectations.**[12]

If all investors have homogeneous expectations, Figure 9.8 would be the same for all individuals. That is, all investors would sketch out the same efficient set of risky assets because they would be working with the same inputs. This efficient set of risky assets is represented by the curve *XAY*. Because the same risk-free rate would apply to everyone, all investors would view point A as the portfolio of risky assets to be held.

This point A takes on great importance because all investors would purchase the risky securities that it represents. Those investors with a high degree of risk-aversion might combine A with an investment in the riskless asset, achieving point *4*, for example. Others with low aversion to risk might borrow to achieve, say, point *5*. Because this is a very important conclusion, we restate it:

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[12] The assumption of homogeneous expectations states that all investors have the same beliefs concerning returns, variances, and covariances. It does not say that all investors have the same aversion to risk.

In a world with homogeneous expectations, all investors would hold the portfolio of risky assets represented by point *A*.

If all investors choose the same portfolio of risky assets, it is possible to determine what that portfolio is. We show below that it is the **market portfolio.**

▪ EXAMPLE

Imagine a stock market composed of only four securities: Alpha Electronics, Beta Food Products, Gamma Clothing, and Delta Home Builders. Financial details on the four securities are provided in Table 9.4.

Further imagine that there are only three investors holding stocks. All of these investors have homogeneous expectations. The investors and their wealth in stocks are

	Wealth in stock market	Ownership of market (percent)
Mary Russell	$50,000,000	50
Vincent Dinoso	49,980,000	49.98
Walter Peck	20,000	0.02
Total	$100,000,000	100

Under the assumption that all three individuals have homogeneous expectations, we know from our earlier analysis that all investors must be holding identical portfolios of risky assets. Suppose that each investor holds shares in a security based on the percentage of the market he or she owns. That is

$$\begin{array}{c}\text{Shares}\\ \text{of a security}\\ \text{held by an}\\ \text{individual}\end{array} = \begin{array}{c}\text{Percentage}\\ \text{of the entire}\\ \text{market that the}\\ \text{individual owns}\end{array} \times \begin{array}{c}\text{Number}\\ \text{of shares}\\ \text{of security}\\ \text{outstanding}\end{array} \qquad (9.13)$$

For example, Walter Peck owns 0.02% of the market. Thus, he would hold

600 shares (0.02% × 3 million shares) of Alpha Electronics
500 shares (0.02% × 2.5 million shares) of Beta Food Products

Table 9.4 Financial details of securities in the market place

(1) Name	(2) Price per share	(3) Number of shares outstanding	(4) Total* market value	(5) Percentage of market
Alpha Electronics	$ 5	3,000,000	$15,000,000 = $5 × 3,000,000	15% = $\frac{\$15,000,000}{\$100,000,000}$
Beta Food Products	$10	2,500,000	$25,000,000	25
Gamma Clothing	$20	2,000,000	$40,000,000	40
Delta Home Builders	$40	500,000	$20,000,000	20
			$100,000,000	100

* Total market value = Price per share × Number of shares outstanding.

Table 9.5 Holdings of each investor in different stocks assuming that each investor takes a position in a stock proportionate to the number of the stock's outstanding shares

	Mary Russell		*Vincent Dinoso*		*Walter Peck*		*Total*	
	(1)	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*	*(8)*
	Shares	Dollar value	Shares	Dollar value	Shares	Dollar value	Shares held by all investors	Dollar value of stock held by all investors
Alpha Electronics	1,500,000	$7.5 million	1,499,400	$ 7.497 million	600	$3,000	3 million*	$15 million†
Beta Food Products	1,250,000	$12.5 million	1,249,500	$12.495 million	500	$5,000	2.5 million	$25 million
Gamma Clothing	1,000,000	$20 million	999,600	$19.992 million	400	$8,000	2 million	$40 million
Delta Home Builders	250,000	$10 million	249,900	$ 9.996 million	100	$4,000	0.5 million	$20 million
Total		$50 million		$49.98 million		$20,000		$100 million

* 3 million = 1,500,000 + 1,499,400 + 600
† $15 million = $7.5 million + $7.497 million + $3,000

400 shares (0.02% × 2 million shares) of Gamma Clothing
100 shares (0.02% × 0.5 million shares) of Delta Home Builders

This strategy is often called ***holding the market portfolio*** because the individual is holding a constant percentage of the number of outstanding shares in each security in the marketplace.[13] Table 9.5 represents this strategy for each of the three investors. A comparison of column (7) in Table 9.5 with column (3) of Table 9.4 shows that under this strategy investors in aggregate hold *exactly* the number of outstanding shares of each security. In other words, financial economists say that the market *clears*.

Alternatively, suppose that all investors want to hold the same non-market portfolio. For example, suppose that, because Alpha is selling at a low price, each investor wants to hold more shares of Alpha Electronics than the number determined by formula (9.13). Investors want more shares of Alpha than are outstanding, thus the market does not clear. Similarly if each investor wants to hold fewer shares of Delta Home Builders than determined by (9.13) because Delta is selling at a high price, they want fewer shares of Delta than are outstanding, and the market does not clear. In equilibrium, the price of Alpha must rise and the price of Delta must fall until investors willingly hold the exact number of outstanding shares. The market then clears. This would occur only when each investor holds the market portfolio. Thus, we have a very important result:

In a world of homogeneous expectations, the market will only clear when each investor holds the market portfolio.

[13] Perhaps it would be more correct to say each investor holds a portion or percentage of the market portfolio.

Definition of Risk When Investors Hold the Market Portfolio

The previous section shows that all individuals must hold the market portfolio in a world with homogeneous expectations. This result allows us to be more precise about the risk of a security in the context of a diversified portfolio. We represent the variance of a portfolio as the sum of all the boxes in Table 9.6. Table 9.6 is identical to Table 9.1 except for shadings to emphasize security 2.

Consider security 2 in the matrix. What is the contribution of this security to the risk of the entire portfolio? We can answer this question by examining the shaded row in Table 9.6. As indicated by the shaded area, security 2 appears in each box in the row. The row contains the covariances between security 2 and all of the securities in the portfolio, including the covariance of security 2 with itself. (The covariance of security 2 with itself, $\text{Cov}(R_2, R_2)$, is $\text{Var}(R_2) \equiv \sigma_2^2$ by definition.) That is, the row can be written as

$$= X_2X_1\text{Cov}(R_2, R_1) + X_2^2\text{Cov}(R_2, R_2) + X_2X_3\text{Cov}(R_2, R_3) + \cdots + X_2X_N\text{Cov}(R_2, R_N)$$

$$= X_2[X_1\text{Cov}(R_2, R_1) + X_2\text{Cov}(R_2, R_2) + X_3\text{Cov}(R_2, R_3) + \cdots + X_N\text{Cov}(R_2, R_N)]$$

$$= X_2\begin{bmatrix}\text{Contribution of security 2 to the risk of the entire portfolio,} \\ \text{standardized by the percentage of security 2 in the portfolio}\end{bmatrix} \qquad (9.14)$$

One can argue that (9.14) measures the contribution of security 2 to the risk of the portfolio.

The second row of (9.14) factors out X_2, the percentage of security 2 in the entire portfolio. Clearly, the contribution of security 2 to the risk of the entire portfolio is proportional to X_2, the percentage of security 2 in the entire portfolio. Thus, the terms in (9.14) within brackets can be viewed as the contribution of security 2 to the risk of the entire portfolio, *standardized* by the percentage of

Table 9.6 Matrix used to calculate the variance of a portfolio

Stock	1	2	3	. . .	N
1	$X_1^2\sigma_1^2$	$X_1X_2\text{Cov}(R_1, R_2)$	$X_1X_3\text{Cov}(R_1, R_3)$		$X_1X_N\text{Cov}(R_1, R_N)$
2	$X_2X_1\text{Cov}(R_2, R_1)$	$X_2^2\sigma_2^2$	$X_2X_3\text{Cov}(R_2, R_3)$		$X_2X_N\text{Cov}(R_2, R_N)$
3	$X_3X_1\text{Cov}(R_3, R_1)$	$X_3X_2\text{Cov}(R_3, R_2)$	$X_3^2\sigma_3^2$		$X_3X_N\text{Cov}(R_3, R_N)$
. . .					
N	$X_NX_1\text{Cov}(R_N, R_1)$	$X_NX_2\text{Cov}(R_N, R_2)$	$X_NX_3\text{Cov}(R_N, R_3)$		$X_N^2\sigma_N^2$

This chart is identical to that in Table 9.1 except that we are speaking now of a specific portfolio—the market portfolio. X_i now stands for the proportion of security i in the market portfolio.

security 2 in the portfolio. This is perhaps the best measure of the risk of a security in a large portfolio.[14]

The set of terms in (9.14) within brackets has an interesting interpretation. X_i is equal to the proportion of each individual's portfolio that is invested in security i. We argued that, under homogeneous expectations, all investors hold the market portfolio. Under this assumption of homogeneous expectations, X_i is the ratio of the market value of security i to the value of the entire market. Thus, the terms inside the bracket represent a weighted average of the covariances between security 2 and each security in the market, where the weights are proportional to the market value of each security in the market.

It can be shown that

$$\mathrm{Cov}(R_2, R_M) = X_1\mathrm{Cov}(R_2, R_1) + X_2\mathrm{Cov}(R_2, R_2) + X_3\mathrm{Cov}(R_2, R_3) + \cdots + X_N\,\mathrm{Cov}(R_2, R_N)$$

where $\mathrm{Cov}(R_2, R_M)$ is the covariance between the return on security 2 and the return on the market portfolio. Thus, the standardized contribution of security 2 to the risk of the market portfolio is proportional to the covariance between the return on security 2 and the return on the market as a whole. Of course, there is nothing special about security 2. The contribution of any security i to the risk of the market portfolio, standardized by its percentage in the portfolio, can be represented as

$$\mathrm{Cov}(R_i, R_M) \tag{9.15}$$

Researchers have *further* standardized (9.15) by defining the beta of security i to be[15]

$$\beta_i = \frac{\mathrm{Cov}(R_i, R_M)}{\sigma^2(R_M)} \tag{9.16}$$

---

[14] The explanation we have presented is heuristic in nature. We confess that there has been a sleight of hand at a few crucial points. A more rigorous derivation can be stated as follows. The variance of the portfolio can be represented as

$$\sigma_P^2 = \text{Variance of portfolio} = \sum_{i=1}^{N}\sum_{j=1}^{N} X_i X_j \sigma_{ij} \tag{a}$$

where σ_{ij} is the covariance of i with j if $i \neq j$, and σ_{ij} is the variance or σ_i^2 if $i = j$. The double summation in (a) sums across all boxes in Table 9.6.

The contribution of security i to the risk of a portfolio can be best written as $\partial\sigma_P^2/\partial X_i$. This measures the change in the variance of the entire portfolio when the proportion of security i is increased slightly. For security 2,

$$\frac{\partial\sigma_P^2}{\partial X_2} = 2\sum_{j=1}^{N} X_j\sigma_{i2} = 2[X_1\mathrm{Cov}(R_1, R_2) + X_2\sigma_2^2 + X_3\mathrm{Cov}(R_3, R_2) + \cdots + X_N\,\mathrm{Cov}(R_N, R_2)] \tag{b}$$

The term within brackets in (b) is $\mathrm{Cov}(R_2, R_M)$. Hence we can rewrite (b) as

$$\frac{\partial\sigma_P^2}{\partial X_2} = 2\,\mathrm{Cov}(R_2, R_M) \tag{c}$$

The 2 in (c) occurs because the terms involving security 2 in both the second row and the second column are involved. Though the variance term, $\sigma_{22} = \sigma_2^2$, occurs only once, note that

$$\frac{\partial X_2^2\sigma_2^2}{\partial X_2} = 2X_2\sigma_2^2$$

[15] By noting that $\mathrm{Cov}(R_i, R_M) = \rho_{i,M}\sigma_i\sigma_M$, we can rewrite β_i as

$$\beta_i = \rho_{i,M}\frac{\sigma_i}{\sigma_M}$$

where $\sigma^2(R_M)$ is the variance of the market. Though both $\text{Cov}(R_i, R_M)$ and β_i can be used as measures of the contribution of security i to the risk of the market portfolio, β_i is much more common. One useful property is that the average beta across all securities, when weighted by the proportion of each security's market value to that of the market portfolio, is 1. That is,

$$\sum_{i=1}^{N} X_i \beta_i = 1 \tag{9.17}$$

Beta as a Measure of Responsiveness

The previous discussion shows that the beta of a security is the standardized covariance between the return on the security and the return on the market. Though this explanation is 100% correct, it is not likely to be 100% intuitively appealing to anyone other than a statistician. Luckily, there is a more intuitive explanation for beta. We present this explanation through an example.

EXAMPLE

Consider the following possible returns on both the stock of Jelco, Inc., and on the market:

State	Type of economy	Return on market (percent)	Return on Jelco, Inc. (percent)
I	Bull	15	25
II	Bull	15	15
III	Bear	−5	−5
IV	Bear	−5	−15

Though the return on the market has only two possible outcomes (15% and −5%), the return on Jelco has four possible outcomes. It is helpful to consider the expected return on a security for a given return on the market. Assuming each state is equally likely, we have

Type of economy	Return on market (percent)	Expected return on Jelco, Inc. (percent)
Bull	15%	20% = 25% × ½ + 15% × ½
Bear	−5%	−10% = −5% × ½ + (−15%) × ½

Jelco, Inc., responds to market movements because its expected return is greater in bullish states than in bearish states. We now calculate exactly how responsive the security is to market movements. The market's return in a bullish economy is 20 percent (15% − (−5%)) greater than the market's return in a bearish economy. However, the expected return on Jelco in a bullish economy is 30 percent (20% − (−10%)) greater than its expected return in a bearish state. Thus, Jelco, Inc., has a responsiveness coefficient of 1.5 (30%/20%).

This relationship appears in Figure 9.9. The returns for both Jelco and the market in each state are plotted as four points. In addition, we plot the expected

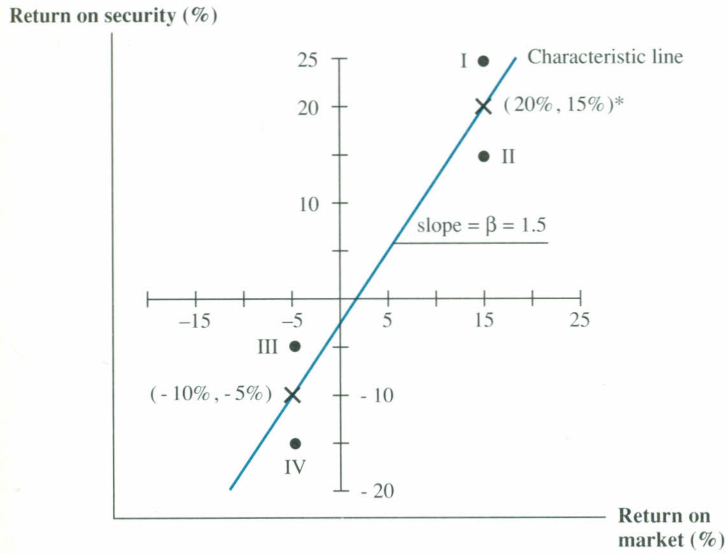

Figure 9.9 Performance of Jelco, Inc., and the market portfolio.

■ The two points marked **X** represent the expected return on Jelco for each possible outcome of the market portfolio. The expected return on Jelco is positively related to the return on the market. Because the slope is 1.5, we say that Jelco's beta is 1.5. Beta measures the responsiveness of the security's return to movement in the market.

* (20%, 15%) refers to the point where the return on the security is 20% and the return on the market is 15%.

return on the security for each of the two possible returns on the market. These two points, which we designate by *X*, are joined by a line called the **characteristic line** of the security. The slope of the line is 1.5, the number calculated in the previous paragraph. This responsiveness coefficient of 1.5 is the beta of Jelco.

The interpretation of beta from Figure 9.9 is intuitive. The graph tells us that the returns on Jelco are magnified 1.5 times over those of the market. When the market does well, Jelco's stock is expected to do even better. When the market does poorly, Jelco's stock is expected to do even worse. Now imagine an individual with a portfolio near that of the market who is considering the addition of Jelco to his portfolio. Because of Jelco's *magnification factor* of 1.5, he will view this stock as contributing much to the risk of the portfolio. We showed earlier that the beta of the average security in the market is 1. Jelco contributes more to the risk of a large, diversified portfolio than does an average security because Jelco is more responsive to movements in the market.

Further insight can be gleaned by examining securities with negative betas. One should view these securities as either hedges or insurance policies. The security is expected to do well when the market does poorly and vice-versa. Because of this, adding a negative beta security to a large, diversified portfolio actually reduces the risk of the portfolio.[16]

[16] Unfortunately, empirical evidence shows that virtually no stocks have negative betas.

A Test

We have put this question on past corporate finance examinations:

1. What sort of investor rationally views the variance (or standard deviation) of an individual security's return as the security's proper measure of risk?
2. What sort of investor rationally views the beta of a security as the security's proper measure of risk?

A proper answer might be something like the following:

> A rational, risk-averse investor views the variance (or standard deviation) of her portfolio's return as the proper measure of the risk of her portfolio. If for some reason or another the investor can hold only one security, the variance of that security's return becomes the variance of the portfolio's return. Hence, the variance of the security's return is the security's proper measure of risk.
>
> If an individual holds a diversified portfolio, she still views the variance (or standard deviation) of her portfolio's return as the proper measure of the risk of her portfolio. However, she is no longer interested in the variance of each individual security's return. Rather, she is interested in the *contribution* of an individual security to the variance of the portfolio.

Under the assumption of homogeneous expectations, all individuals hold the market portfolio. Thus, we measure risk as the contribution of an individual security to the variance of the market portfolio. This contribution, when standardized properly, is the beta of the security. While very few investors hold the market portfolio exactly, many hold reasonably diversified portfolios. These portfolios are close enough to the market portfolio so that the beta of a security is likely to be a reasonable measure of its risk.

Concept Questions

- If all investors have homogeneous expectations, what portfolio of risky assets do they hold?
- What is the formula for beta?
- Why is beta the appropriate measure of risk for a single security in a large portfolio?

9.7 Relationship between Risk and Return

It is commonplace to argue that the expected return on a security should be positively related to its risk. That is, individuals will hold a risky security only if its expected return compensates for its risk. This reasoning holds regardless of the measure of risk. Now consider our world where all individuals (1) have homogeneous expectations and (2) all individuals can borrow and lend at the risk-free rate. All individuals hold the market portfolio of risky securities here. We have shown that the beta of a security is the appropriate measure of risk in this context. Hence,

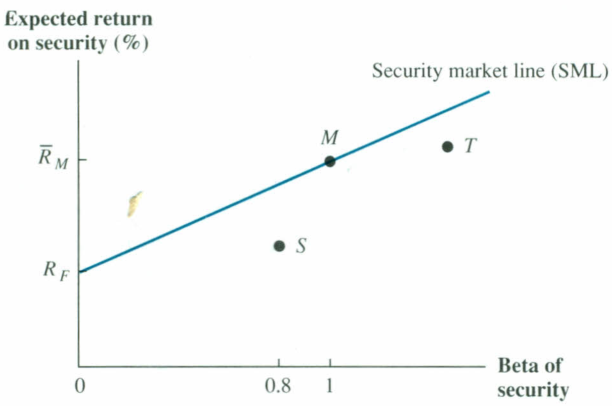

Figure 9.10 Relationship between expected return on an individual security and beta of the security.

R_F is the risk-free rate.

$\overline{R}_M$ is the expected return on the market portfolio.

the expected return on a security should be positively related to its beta. This is illustrated in Figure 9.10. The upward-sloping line in the figure is called the **security-market line** (SML).

There are six important points associated with this figure.

1. *A beta of zero.* The expected return on a security with a beta of zero is the risk-free rate, R_F. Because a security with zero beta has no relevant risk, its expected return should equal the risk-free rate.

2. *A beta of one.* Equation (9.17) points out that the average beta across all securities, when weighted by the proportion of each security's market value to that of the market portfolio, is 1. Because the market portfolio is formed by weighting each security by its market value, the beta of the market portfolio is 1. Because all securities with the same beta have the same expected return, the expected return for any security with a beta of 1 is $\overline{R}_M$, the expected return on the market portfolio.

3. *Linearity.* The intuition behind an upwardly sloping curve is clear. Because beta is the appropriate measure of risk, high-beta securities should have an expected return above that of low-beta securities. However, Figure 9.10 shows something more than an upwardly sloping curve; the relationship between expected return and beta corresponds to a *straight* line.

It is easy to show that the line in Figure 9.10 is straight. To see this, consider security *S* with, say, a beta of 0.8. This security is represented by a point below the security-market line in the figure. Any investor could duplicate the beta of security *S* by buying a portfolio with 20 percent in the risk-free asset and 80 percent in a security with a beta of 1. However, the homemade portfolio would itself lie on the SML. In other words, the portfolio dominates security *S* because the portfolio has a higher expected return and the same beta.

Now consider security *T* with, say, a beta greater than 1. This security is also below the SML in Figure 9.10. Any investor could duplicate the beta of security *T* by borrowing to invest in a security with a beta of 1. This portfolio must also lie on the SML, thereby dominating security *T*.

Because no one would hold either *S* or *T*, their stock prices would drop. This price adjustment would raise the expected returns on the two securities. The price adjustment would continue until the two securities lay on the security-market line.

The above example considered two overpriced stocks and a straight SML. Securities lying above the SML are *underpriced.* Their prices must rise until their expected returns lie on the line. If the SML is itself curved, many stocks would be mispriced. In equilibrium, all securities would be held only when prices changed so that the SML became straight. In other words, linearity would be achieved.

4. *The Capital-Asset-Pricing Model.* You may remember from algebra courses that a line can be described algebraically if one knows both its intercept and its slope. We can see from Figure 9.10 that the intercept of the SML is R_F. Because the expected return of any security with a beta of 1 is $\bar{R}_M$, the slope of the line is $\bar{R}_M - R_F$. This allows us to write the SML algebraically as

Capital-asset-pricing model:

$$\bar{R} = R_F + \beta (\bar{R}_M - R_F) \qquad (9.18)$$

Expected return on a security	=	Risk free rate	+	Beta of the security	×	Difference between expected return on market and risk-free rate

According to financial economists, the above algebraic formula describing the SML is called the **capital-asset-pricing model.** The formula can be illustrated by assuming a few special cases.

a. Assume that $\beta = 0$. Here, $\bar{R} = R_F$, that is, the expected return on the security is equal to the risk-free rate. We argued this in point (1) above.
b. Assume $\beta = 1$. The equation reduces to $\bar{R} = \bar{R}_M$, that is, the expected return on the security is equal to the expected return on the market. We argued this in point (2) above.

As with any line, the line represented by equation (9.18) has both a slope and an intercept. R_F, the risk-free rate, is the intercept. Because the beta of a security is the horizontal axis, $\bar{R}_M$ less R_F is the slope. The line will be upward-sloping as long as the expected return on the market is greater than the risk-free rate. Because the market portfolio is a risky asset, theory suggests that its expected return is above the risk-free rate. In addition, the empirical evidence of the previous chapter showed that the actual return on the market portfolio over the past 63 years was well above the risk-free rate.

■ EXAMPLE

The stock of Aardvark Enterprises has a beta of 1.5 and that of Zebra Enterprises has a beta of 0.7. The risk-free rate is 7 percent and the difference between the expected return on the market and the risk-free rate is 8.5 percent.[17] The expected returns on the two securities are

[17] As reported in Table 8.2, Ibbotson and Sinquefield found that the expected return on common stocks was 12.1 percent over 1926–1988. The average risk-free rate over the same time interval was 3.6 percent. Thus, the average difference between the two was 8.5 percent (12.1% − 3.6%). Financial economists use this as the best estimate of the difference to occur in the future. We will use it frequently in this text.

Expected return for Aardvark:

$$19.75\% = 7\% + 1.5 \times 8.5\% \tag{9.18'}$$

Expected return for Zebra:

$$12.95\% = 7\% + 0.7 \times 8.5\%$$

5. *Portfolios as well as securities.* Our discussion of the CAPM considered individual securities. Does the relationship in Figure 9.10 and equation (9.18) hold for portfolios as well?

Yes. To see this, consider a portfolio formed by investing equally in our two securities, Aardvark and Zebra. The expected return on the portfolio is

Expected return on portfolio:

$$16.35\% = 0.5 \times 19.75\% + 0.5 \times 12.95\% \tag{9.19}$$

The beta of the portfolio is simply a weighted average of the two securities. Thus we have

Beta of portfolio:

$$1.1 = 0.5 \times 1.5 + 0.5 \times 0.7$$

Under the CAPM, the expected return on the portfolio is

$$16.35\% = 7\% + 1.1 \times 8.5\% \tag{9.20}$$

Because the value in (9.19) is the same as the value in (9.20), the example shows that the CAPM holds for portfolios as well as for individual securities.

6. *A potential confusion.* Students often confuse the SML in Figure 9.10 with the capital-market line (line *II* in Figure 9.8). Actually, the lines are quite different. The capital-market line traces the efficient set of portfolios formed from both risky assets and the riskless asset. Each point on the line represents an entire portfolio. Point *A* is a portfolio composed entirely of risky assets. Every other point on the line represents a portfolio of the securities in *A* combined with the riskless asset. The axes on Figure 9.8 are expected return of a *portfolio* and the standard deviation of a *portfolio.* Individual securities do not lie along line *II*.

The SML in Figure 9.10 relates expected return to beta. Figure 9.10 differs from Figure 9.8 in at least two ways. First, beta appears in the horizontal axis of Figure 9.10 but standard deviation appears in the horizontal axis of Figure 9.8. Second, the SML in Figure 9.10 holds both for all individual securities and for all possible portfolios, whereas line *II* (the capital-market line) in Figure 9.8 holds only for efficient portfolios.

Concept Questions

- Why is the SML a straight line?
- What is the capital-asset-pricing model?
- What are the differences between the capital-market line and the security-market line?

9.8 Summary and Conclusions

This chapter sets forth the fundamentals of modern portfolio theory. Our basic points are these:

1. The previous chapter showed us how to calculate the expected return and variance for individual securities, and the covariance and correlation for pairs of securities. Given these statistics, the expected return and variance for a portfolio of two securities A and B can be written as

$$\text{Expected return on portfolio} = X_A\overline{R}_A + X_B\overline{R}_B$$
$$\text{Var(portfolio)} = X_A^2\sigma_A^2 + 2X_AX_B\sigma_{AB} + X_B^2\sigma_B^2$$

2. In our notation, X stands for the proportion of a security in one's portfolio. By varying X, one can trace out the efficient set of portfolios. We graphed the efficient set for the two-asset case as a curve, pointing out that the degree of curvature or bend in the graph reflects the diversification effect: The lower the correlation between the two securities, the greater the bend. Without proof, we stated that the general shape of the efficient set holds in a world of many assets.
3. Just as the formula for variance in the two-asset case is computed from a 2×2 matrix, the variance formula is computed from an $N \times N$ matrix in the N-asset case. We show that, with a large number of assets, there are many more covariance terms than variance terms in the matrix. In fact, the variance terms are effectively diversified away in a large portfolio but the covariance terms are not. Thus, a diversified portfolio can only eliminate some, but not all, of the risk of the individual securities.
4. The efficient set of risky assets we spoke of earlier can be combined with riskless borrowing and lending. In this case, a rational investor will always choose to hold the portfolio of risky securities represented by point A in Figure 9.8, then can either borrow or lend at the riskless rate to achieve any desired point on the capital-market line.
5. If (1) all investors have homogeneous expectations and (2) all investors can borrow and lend at the riskless rate, all investors will choose to hold the portfolio of risky securities represented by point A. They will then either borrow or lend at the riskless rate. In a world of homogeneous expectations, point A represents the market portfolio.
6. The contribution of a security to the risk of a large portfolio is the sum of the covariances of the security's return with the returns on the other securities in the portfolio. The contribution of a security to the risk of the market portfolio is the covariance of the security's return with the market's return. This contribution, when standardized, is called the beta. The beta of a security can also be interpreted as the responsiveness of a security's return to that of the market.
7. The CAPM states that

$$\overline{R} = R_F + \beta(\overline{R}_M - R_F)$$

In other words, the expected return on a security is positively (and linearly) related to the security's beta.

Key Terms

Portfolio, *256*
Opportunity (feasible) set, *263*
Efficient set, *264*
Systematic (market) risk, *272*
Diversifiable (unique) (unsystematic) risk, *272*
Risk-averse, *272*
Capital market line, *276*
Separation principle, *276*
Homogeneous expectations, *277*
Market portfolio, *278*
Characteristic line, *283*
Security market line, *285*
Capital-asset-pricing model, *286*

Suggested Readings

The capital-asset-pricing model was originally published in these two classic articles:

Sharpe, W. F. "Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk." *Journal of Finance* (September 1964).

Lintner, J. "Security Prices, Risk and Maximal Gains from Diversification." *Journal of Finance* (December 1965).

Questions and Problems

9.1 A portfolio consists of 20 shares of Andrews stock which sells for $50 per share and 30 shares of Dean stock which sells for $20 per share. What are the weights of the two stocks in this portfolio?

9.2 Security *F* has an expected return of 10% and a standard deviation of 5% per year. Security *G* has an expected return of 20% and a standard deviation of 60% per year. The correlation between *F* and *G* is 0.5.

a. What is the expected return on a portfolio composed 40% of security *F* and 60% of security *G*?

b. What is the standard deviation of this portfolio?

9.3 Suppose the expected returns and variances of stocks *A* and *B* are $\overline{R}_A = 0.2$, $\overline{R}_B = 0.3$, $\sigma_A^2 = 0.1$, and $\sigma_B^2 = 0.2$, respectively.

a. Calculate the expected return and variance of a portfolio that is composed of 60% *A* and 40% *B* when the correlation coefficient between the stocks is −0.5.

b. Calculate the expected return and variance of a portfolio that is composed of 60% *A* and 40% *B* when the correlation coefficient between the stocks is −0.6.

c. How does the correlation coefficient affect the variance of the portfolio?